The Dozenal Society of America

Analysis of Multiplication Tables

by Michael De Vlieger

In the charts on the left, trends in the end digits of the multiplication facts for each digit are illustrated. Each triangular trend shape (a "phase") begins at the point where the end digit equals zero, then expands to the full width of the column as the end digits of each fact increase. In the octal example for the products of 2, the phase begins at zero and grows in thickness as we proceed from 2 to 4 to 6, until, at "10", the phase is widest. At the 4th product, $2 \times 4 = 10_{\text{EIGHT}}$. This and any other instance of a multiple of the base are circled. The trends for end digits of products for digits n smaller than r/2 are increasing. The end digits increment by the digit, until they equal r. When they reach r, they complete a period whose length is r/n. The trend for the end digits of products for digits n > r/2 is decreasing. The end digits appear to begin at larger values and decrease as the multiplier increases. Trends associated with divisors are colored red with a thick solid edge. Those of semidivisors are colored orange, with a ragged edge. Trends for semitotatives are yellow with a ragged edge. Those of digits coprime to the base are gray. Coprime digits which are divisors of (r – 1) are colored light blue. For even bases, the products of the divisor r/2 have a phase shape that looks different from the triangular phases greater than or lesser than r/2 (see 5 in decimal, 6 in dozenal, etc.). This is because, in the multiplication table, the end digits of products of r/2 seem to alternate between 0 and r/2.

Octal (Base 8)

1	2	3	4	5	6	7	10
2	4	6	10	12	14	16	20
3	6	11	14	17	22	25	30
4	10	14	20	24	30	34	40
5	12	17	24	31	36	43	50
6	14	22	30	36	44	52	60
7	16	25	34	43	52	61	70
10	20	30	40	50	60	70	100

Decimal (Base 10)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Dozenal (Base 12)

Numeral Set:

DECIMAL EQUIVALENT											
0	1	2	3	4	5	6	7	8	9	10	11
0	1	2	3	4	5	6	7	8	9	χ	Ɛ
DOZENAL DIGITS											

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Octal (Base 8)

1	2	3	4	5	6	7	10
2	4	6	10	12	14	16	20
3	6	11	14	17	22	25	30
4	10	14	20	24	30	34	40
5	12	17	24	31	36	43	50
6	14	22	30	36	44	52	60
7	16	25	34	43	52	61	70
10	20	30	40	50	60	70	100

Decimal (Base 10)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Dozenal (Base 12)

Numeral Set:

DECIMAL EQUIVALENT											
0	1	2	3	4	5	6	7	8	9	10	11
0	1	2	3	4	5	6	7	8	9	χ	Ɛ
DOZENAL DIGITS											

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Note that there are no standard dozenal numerals. The numerals presented here are the set of Dwiggins duodecimal numerals, used by the DSA between 1945 and 1974, then restored in 2008. Other numerals are proposed by other organizations and individuals.

Hexadecimal (Base 16)

Numeral Set:

DECIMAL EQUIVALENT	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEXADECIMAL DIGITS	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f

1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10
2	4	6	8	a	c	e	10	12	14	16	18	1a	1c	1e	20
3	6	9	c	f	12	15	18	1b	1e	21	24	27	2a	2d	30
4	8	c	10	14	18	1c	20	24	28	2c	30	34	38	3c	40
5	a	f	14	19	1e	23	28	2d	32	37	3c	41	46	4b	50
6	c	12	18	1e	24	2a	30	36	3c	42	48	4e	54	5a	60
7	e	15	1c	23	2a	31	38	3f	46	4d	54	5b	62	69	70
8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	80
9	12	1b	24	2d	36	3f	48	51	5a	63	6c	75	7e	87	90
a	14	1e	28	32	3c	46	50	5a	64	6e	78	82	8c	96	a0
b	16	21	2c	37	42	4d	58	63	6e	79	84	8f	9a	a5	b0
c	18	24	30	3c	48	54	60	6c	78	84	90	9c	a8	b4	c0
d	1a	27	34	41	4e	5b	68	75	82	8f	9c	a9	b6	c3	d0
e	1c	2a	38	46	54	62	70	7e	8c	9a	a8	b6	c4	d2	e0
f	1e	2d	3c	4b	5a	69	78	87	96	a5	b4	c3	d2	e1	f0
10	20	30	40	50	60	70	80	90	a0	b0	c0	d0	e0	f0	100

This hexadecimal multiplication table uses standard alphanumeric digits.

Hexadecimal (Base 16)

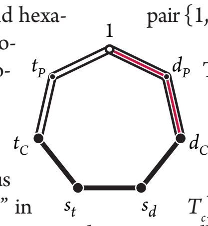
Numeral Set:

DECIMAL EQUIVALENT	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
HEXADECIMAL DIGITS	0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f

1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10
2	4	6	8	a	c	e	10	12	14	16	18	1a	1c	1e	20
3	6	9	c	f	12	15	18	1b	1e	21	24	27	2a	2d	30
4	8	c	10	14	18	1c	20	24	28	2c	30	34	38	3c	40
5	a	f	14	19	1e	23	28	2d	32	37	3c	41	46	4b	50
6	c	12	18	1e	24	2a	30	36	3c	42	48	4e	54	5a	60
7	e	15	1c	23	2a	31	38	3f	46	4d	54	5b	62	69	70
8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	80
9	12	1b	24	2d	36	3f	48	51	5a	63	6c	75	7e	87	90
a	14	1e	28	32	3c	46	50	5a	64	6e	78	82	8c	96	a0
b	16	21	2c	37	42	4d	58	63	6e	79	84	8f	9a	a5	b0
c	18	24	30	3c	48	54	60	6c	78	84	90	9c	a8	b4	c0
d	1a	27	34	41	4e	5b	68	75	82	8f	9c	a9	b6	c3	d0
e	1c	2a	38	46	54	62	70	7e	8c	9a	a8	b6	c4	d2	e0
f	1e	2d	3c	4b	5a	69	78	87	96	a5	b4	c3	d2	e1	f0
10	20	30	40	50	60	70	80	90	a0	b0	c0	d0	e0	f0	100

This hexadecimal multiplication table uses standard alphanumeric digits.

These analyses of the patterns in the octal, decimal, dozenal, and hexadecimal multiplication tables explore the impact of the prime composition of the number base on the ease with which one might memorize the multiplication tables of that base.



Each positive integer r considered as a number base possesses r digits, zero through $(r-1)$. Any one of these digits we can consider n. The base r has a modular-math relationship to each of its digits; thus we can interpret the digit "0" as congruent to r. Hence, the digit "0" in decimal is congruent with the integer χ, while in hexadecimal, it is congruent with 14;. There are seven types of relationships r can have with each of its digits. (Here we are only considering positive, integral bases between 8 and 14;)

Oftentimes we read that divisors, those digits D which divide r such that the resulting quotient is also an integral divisor D', are key tools in the usefulness of the base r. Indeed, divisors in the above charts have periods of integral length, that is, they return to a multiple of r after a period of D' products in the multiplication table. This is part of what makes the multiplication tables of some bases easier to memorize than others.

There is also a sort of "resistance" which often gets ignored, but after r = 26; it dominates. This "resistance" is the set of "totative" digits T which are coprime to r, meaning that the least common multiple between r and T is $(r \times T)$. These digits repeat only after the full length r of the multiplication table line they govern. Seven, coprime to all the bases analyzed here, features facts in the table which only repeat after r products, thus for octal, after 8 products, at octal 70, for decimal after χ facts, at decimal 70. Only four bases {2, 4, 6, 10;} have divisor-dominant digit ranges; an additional six {3, 8, χ, 16;, 20;, 26;} enjoy parity between divisors and coprime digits. For all other bases, stubborn totative "resistance" is the stronger force, not that of happy divisors.

The number 1 is a special case. Because $(1 \times r) = r$, it is a divisor. Because the least common multiple of 1 and r is $(1 \times r)$, it is coprime to r. Thus it is both a divisor and coprime, a member of both sets D and T.

Each base r has a prime factorization wherein prime numbers, through multiplication, produce r. The prime factorization of 8 = $\{2^3\}$, χ = $\{2 \cdot 5\}$, 10; = $\{2^2 \times 3\}$, 14; = $\{2^4\}$. The distinct prime factors d_p serve as those digits which themselves are prime numbers and also are divisors of r. There may be composite divisors d_c which are both composite numbers, themselves products of some or all of the d_p of r, and are divisors of r. Thus, the set $\{1, D_p, D_c\}$ are divisors of r. The set of digits $\{D_p, D_c\}$ enjoy a regularity in the multiplication table, and regularity in their fractions. The divisor pair $\{1, r\}$ are called "trivial divisors"; every integer possesses such divisors.

Like divisors, coprime digits T can be prime (T_p) or composite (T_c). The prime totatives t_p are prime numbers less than r which are unrepresented in the prime factorization of r. For some bases where certain unrepresented primes are small enough to square (9 in decimal and in hexadecimal) or multiply with another unrepresented prime digit (13; in hexadecimal), composite totatives t_c appear. Thus the set $\{1, T_p, T_c\}$ are coprime to r. The coprime digits in the set $\{T_p, T_c\}$ feature difficult maximally long cycles in the multiplication table, and recurring fractions. The totative pair $\{1, (r-1)\}$ are common to all integral bases—in binary, the digit 1 serves the role of both elements in this pair. The patterns in the multiplication table for the "ω-totative" $(r-1)$ and its factors t_ω are not difficult to memorize; the pattern associated with 1 is trivial. The "ω-totative" $(r-1)$ whose factorization contains unrepresented primes imparts simple digit-sum divisibility rules in base r for those primes.

Some digits S are neither divisors nor coprime to r. Such mixed digits are always composite, either composed of divisors (S_d) or involving both divisors and totatives (S_t). The semidivisors s_d, such as 4 in decimal or 8 in decimal and dozenal, feature more or less regular patterns in the multiplication table and terminating but not single-digit fractions. The semitotatives s_t also exhibit more or less regular multiplication patterns, but their fractions recur after an initial group of digits.

Thus, both the ease of memorization and patterning of the multiplication table of a given base, and the behavior of digital fractions in that base, are effects of the base's prime factors. The more divisors (thus the larger the base) may not necessarily be the road to the best number base. The best base may be determined by number-theoretical considerations as well as the human ability to efficiently compute.

r	1	D_p	D_c	S_d	S_t	T_c	T_p	1
8	1	2	4, 0	—	6	—	3, 5, 7	1
χ	1	2, 5	0	4, 8	6	9	3, 7	1
10	1	2, 3	4, 6, 0	8, 9	χ	—	5, 7, Ɛ	1
14	1	2	4, 8, 0	—	6, χ, 10, 12	9, 13	3, 5, 7, Ɛ, 11	1

The digits of r *sorted according to digit class. Digits on the left may be seen as more friendly, while those on the right tend to act against us when using a number base as a tool.*