The Dozenal Society of America

Dozenal FAQs

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The following ten questions were sent in recently by a student. These lucid questions are some of those most frequently asked. The Dozenal Society of America is furnishing the full responses to these questions complete with illustrations and references. Thanks to Mr. Israeli for questions 1–10!

0. *What is the number one reason why one believes base 12 is superior to decimal?*

A. Briefly, twelve is a highly divisible yet compact number; it has more divisors than ten. This facilitates learning and using arithmetic, and simplifies the natural fractions [1,2,3]. This observation dates as far back as the middle ages, involving luminaries of mathematics such as Blaise Pascal [4,5] and Laplace [6]. Dozens have been used since ancient times; the Roman usage as "*unciæ*"[7,8,9] strongly affecting systems of currency (12 pence to a shilling) and measure (12 inches to the foot, 12 original ounces to a pound, 'inch' and 'ounce' both deriving from Roman *uncia*.)

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 0.1. The decimal multiplication table.

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Figure 0.2. The duodecimal multiplication table.

The integer twelve has six divisors {1, 2, 3, 4, 6, 12} while the number ten only has four {1, 2, 5, 10}. Divisors are important because they exhibit relatively brief and predictable patterns in arithmetic in a number base they divide. We can see this when we look at the decimal multiplication products for the number 5. Decimal products of the number 5 and an integer x have a least significant digit that is one of {0, 5}. Because of this, we understand any integer ending with -0 or -5 is some multiple x of 5. Decimal is also an even base: because of this, the multiplication products of 2 end in one of {0, 2, 4, 6, 8}. In decimal, we can test for evenness and divisibility by 5 simply by examining the last digit of an integer.

The number twelve has two additional divisors than ten. In duodecimal, the number 6 behaves much like 5 in the multiplication table. In duodecimal, any product of 6 and an integer x will have a least significant digit that is one of {0, 6}. Like decimal, duodecimal is even, so any product of 2 and an integer x will end with one of {0, 2, 4, 6, 8, χ}.

Because twelve has two more divisors, users of duodecimal enjoy two other divisor product lines in the multiplication table. Any product $3x$ will end in one of {0, 3, 6, 9}, and any product $4x$ will end in one of {0, 4, 8}.

Because twelve has six divisors, with the smallest four consecutive, it presents a multiplication table featuring brief patterns in the product lines of many numbers. (See page 4 for a study of patterns in the decimal and duodecimal multiplication tables and Appendix A for bases $8 \le r \le 16$.)

The divisors of a base have a second important effect. Every divisor d of base r has a complement d' [10] such that

(0.1) $$r = d \times d'$$

This relationship between a divisor and its complement can be seen in the multiplication table: the decimal products of five have 2 possible end digits (10 = 5 × 2), the duodecimal products of four have three possible end digits (12 = 4 × 3). The relationship also appears in digital expansions of fractions. Generally,

(0.2) $$1/d = d'/r$$

thus we will see the digit d' following the radix point. In base ten we have ½ = 0.5 and ⅕ = 0.2. In base twelve we will have ½ = 0.6, ⅓ = 0.4, ¼ = 0.3 and ⅙ = 0.2. Thus duodecimal features single-place terminating fractions for all the 'natural fractions'[11,12]:

	½	⅓	⅔	¼	¾
Decimal	.5	.333…	.666…	.25	.75
Duodecimal	.6	.4	.8	.3	.9

Figure 0.3. The natural fractions in decimal and duodecimal notation.

One might raise the question, "granted, twelve has more divisors than ten, but what about the other numbers?" First, let's acknowledge that number bases fit for general human computation must lie in a certain range. If a number base is too small, we'll need many digits to express even a relatively small quantity. Imagine a binary speed limit sign said "1000000 kph"! A driver would need to slow down and peruse the sign to be sure not to be speeding at double or bumbling at half the posted limit! The seven decimal-digit telephone number would become nearly a two-dozen-bit string of 1's and 0's. It would be easy to miss out on a hot date if one made a transcription error somewhere. If a number base is too large, then one would need to memorize a larger multiplication table, for example, in order to perform arithmetic. If the Mayans multiplied using modern techniques, would they need to spend more than double the time a decimal society would in mastering their multiplication tables? There are 20 digits in the vigesimal (base-20) system, each with 20 combinations. There are 55 unique products in the decimal multiplication table vs. 210 in the vigesimal and 78 in the duodecimal.

Let's presume a number base will need to be between about 7 or 8 through about 16, possibly including 18 and 20. Here is a list of divisor counts and ratios [13, 14] for bases 2 through 20:

r	$\sigma_0(r)$	$\sigma_0(r)/r$
2	2	100
3	2	66 ⅔
4	3	75
5	2	40
6	4	66 ⅔
7	2	28 4⁄7
8	4	50
9	3	33 ⅓
10	4	40

r	$\sigma_0(r)$	$\sigma_0(r)/r$
11	2	18 2⁄11
12	6	50
13	2	15 5⁄13
14	4	28 4⁄7
15	4	26 ⅔
16	5	31 ¼
17	2	11 15⁄17
18	6	33 ⅓
19	2	10 10⁄19
20	6	30

Figure 0.4. Number of divisors $\sigma_0(r)$ and divisor density $\sigma_0(r)/r$ for each base r between 2 and 20 inclusive.

Note that twelve is the smallest number with 6 divisors, and the largest number with 50% or more of its digits as divisors. Mathematicians refer to the number twelve as a "highly composite number", a number that sets a record number of divisors, i.e., having more divisors than all numbers smaller than it [15, 16]. Further, twelve is the third smallest superior highly composite number [17], part of the sequence [18] of such numbers {2, 6, 12, 60, 120, 360, 2520, … }. Essentially, a SHCN has a greater divisor density than all larger integers.

Twelve gives us the ability to access the power of 6 divisors with a magnitude of number base that is not much larger than decimal. The dozen gives us clean natural fractions, incorporating these into its digital representation. Duodecimal divisors are by far the very most important reason why duodecimal is the optimum number base for general human intuitive computation.

1. *Other then having more factors than 10 does, what other major properties make 12 a better base for a number system?*

A. Briefly, twelve has proportionally fewer totatives (digits that have 1 as the greatest common divisor with the number base) than ten, minimizing resistance to human intuition in computation and manipulation of numbers. Fewer totatives means fewer "difficult" product lines in the multiplication table, fewer occasions of recurrent fractions, a more organized arrangement of primes, and a smaller proportion of other digits and numbers that have recurrent reciprocals. Dozenal has a denser set of regular numbers, aiding human intuition in computation. Regular numbers have terminating digital fractions and divisibility tests akin to that of divisors.

These features are not as easily summarized as the fact that twelve has more divisors than ten, but are just as important. The following pages attempt to explain the brief synopsis above, and prove it using mathematics. The explanation can get technical. The terms used in the above synopsis are explained in the following text. Let's begin by examining the digits of a number base, since many of the properties can be summarized by such an examination.

Five Universal Types of Digits

The digits of a number base are important because a number base "sees" the world through its digits. Let the integer $r \geq 2$ be a number base. A digit is an integer $0 \leq n < r$ [19, 20]. Using digits, we can express the arbitrary integer x as a product of the integers q and r plus the digit n [21, 22, 23]:

$$x = qr + n$$

The digit "0" signifies congruence with the number base; it will be interpreted as signifying r itself as a digit [24]. We know the divisors of ten are {1, 2, 5, 10}; we would interpret the digits {0, 1, 2, 5} as decimal divisor digits. Similarly, the divisors of twelve are {1, 2, 3, 4, 6, 12}, but we would interpret the digits {0, 1, 2, 3, 4, 6} as duodecimal divisor digits.

In general, there are five kinds of digits. Once we've introduced each type, we can examine how the types behave, then arrive at a way that aids the assessment of the merit of all number bases.

The Divisor

We've already read about divisors. We can test whether a digit n is a divisor of r by examining the greatest common divisor (highest common factor) $\gcd(n, r) = n$. In fact, we can distinguish two of the four kinds of digits using $\gcd(n, r)$. Divisors may be prime or composite for a composite number base. All number bases r have the "trivial divisors" $\{1, r\}$ [25].

0	1	2	3	4	5	6	7	8	9

Figure 1.1. The divisors d *of base* r *= 10, shown in red.*

The Totative

Another type of digit n is coprime to base r, meaning that the digit and the base have no common divisors but 1 [25–30]. Thus digit n is coprime to r if and only if $\gcd(n, r) = 1$. Such a digit is called a "totative" [31]. The Euler totient function $\phi(r)$ counts the number of totatives of an integer r [32, 33]. (More will be said about totatives and the Euler totient function later.)

0	1	2	3	4	5	6	7	8	9

Figure 1.2. The totatives t *of base* r *= 10, shown in blue.*

The Unit

The digit 1 is special, since it is both a divisor and a totative of any number base r, as well as neither prime nor composite [34, 35]. Because of this, we can refer to it as a "unit" [36]. Digits that are not units (i.e., digits $n \neq 1$) are referred to as "non-units". Thus we have three kinds of digits we can distinguish using one iteration of the greatest common divisor.

0	1	2	3	4	5	6	7	8	9

Figure 1.3. "Countable" digits ϰ *of base* r *= 10, shown in color.*

Two Kinds of Neutral Digit

[The following is derived from a forthcoming paper. [37]]

If we examine some digits, we will find

$$1 < \gcd(n, r) < r.$$

Since we know a divisor will have $\gcd(n, r) = n$ and a totative will have $\gcd(n, r) = 1$, such a digit n must be neither. Thus there are neutral digits.

0	1	2	3	4	5	6	7	8	9

Figure 1.4. Neutral digits s *of base* r *= 10, shown in gold.*

There are two and only two kinds of "neutral digit". I call these "semidivisors" and "semitotatives". Neutral digits are composite, thus neutral digits are always composite. Further, neutral digits exist only in composite bases $r > 4$.

The Semidivisor, a Regular Number

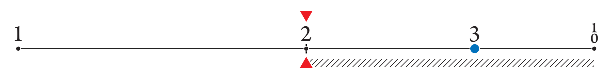
A regular number g in base r is one whose factors are found among the factors of base r [38, 39]. Regular decimal numbers include any power of two or five or any combination of 2 and 5. Examples are the numbers 2, 5, 8, 10, 20, 25, 32, 100, 125, 256, etc. Regular digits are the set of regular numbers less than the base. The set of decimal regular digits is

$$G_{\text{ten}} = \{0, 1, 2, 4, 5, 8\}$$

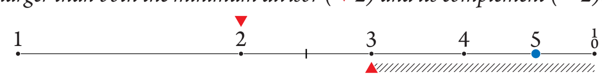
Some of the regular digits are divisors $\{0, 1, 2, 5\}$ but some are not divisors $\{4, 8\}$. The digits $\{4, 8\}$ are neutral digits, since $\gcd(4, 10) = 2$ and $\gcd(8, 10) = 2$. If we examine the prime decompositions of these two digits, we'd see that at least one of the prime factors would have an exponent that exceeds that of the same prime factor in the number base. Indeed, $4 = 2^2$ and $8 = 2^3$, while the prime decomposition of ten is $2 \cdot 5$. The decimal digits $\{4, 8\}$ are semidivisors. Semidivisors exist in composite bases r that are not powers of prime numbers. Base 6 is the smallest base that possesses a semidivisor. Octal and hexadecimal cannot possess semidivisors because these bases are powers of two.

The Semitotative, Product of Divisors & Totatives

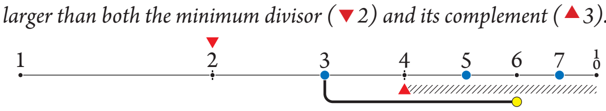
The semitotative is simply a digit that is the product of at least one prime divisor and at least one prime totative. The decimal semitotative is digit 6. In duodecimal, the semitotative is digit ten. In hexadecimal, there are four digits that are semitotatives: $\{6, 10, 12, 14\}$. Semitotatives exist for all composite number bases that have minimum totatives less than the minimum divisor's complement, d'. (See Figure 1.5) Because the minimum totative of bases 4 and 6 are both larger



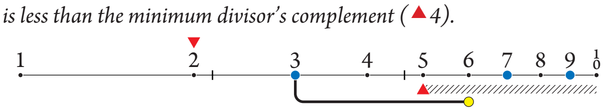
Base 4 cannot have semitotatives because the minimum totative (●3) is larger than both the minimum divisor (▼2) and its complement (▲2).



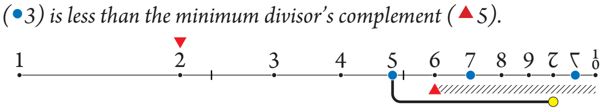
Base 6 cannot have semitotatives because the minimum totative (●5) is larger than both the minimum divisor (▼2) and its complement (▲3).



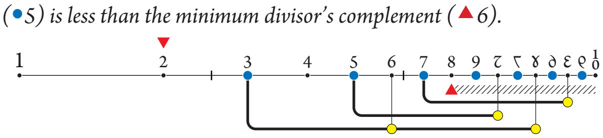
Base 8 possesses the semitotative ●6 because the minimum totative (●3) is less than the minimum divisor's complement (▲4).



Base 10 possesses the semitotative ●6 because the minimum totative (●3) is less than the minimum divisor's complement (▲5).



Base 12 possesses the semitotative ●10 because the minimum totative (●5) is less than the minimum divisor's complement (▲6).



Base 16 has four semitotatives. The totatives ●3, ●5, and ●7 are less than the minimum divisor's complement, ▲8. The totative ●3 generates the semitotatives ●6 and ●12 by multiplication with the divisor ●2 and ●2^2 respectively. The totatives ●5 and ●7 produce the semitotatives ●10 and ●14 respectively with the divisor ●2.

Figure 1.5 ▲

than d', these bases do not have semitotatives. Because number bases like 8 and 16 that are powers of primes don't have semidivisors, semitotatives are their only neutral digits. Base 6 is unique in that it has a semidivisor but no semitotative. As the number base *r* gets larger, semitotatives burgeon. For very large highly composite bases, semitotatives are the most common type of digit, far outweighing the totatives, with vanishingly few regular digits.

Figure 1.6. Semidivisors s_d (orange) and semitotatives s_t (yellow) of base r = 10, shown in gold.

Now we can make a digit map of any number base. In the digit map, totatives are colored a pale gray, as there are several kinds of totative digits.

See Appendix B for digit maps for bases $2 \le r \le 60$. Using these maps, the majority of the properties of the number bases can be determined at a glance.

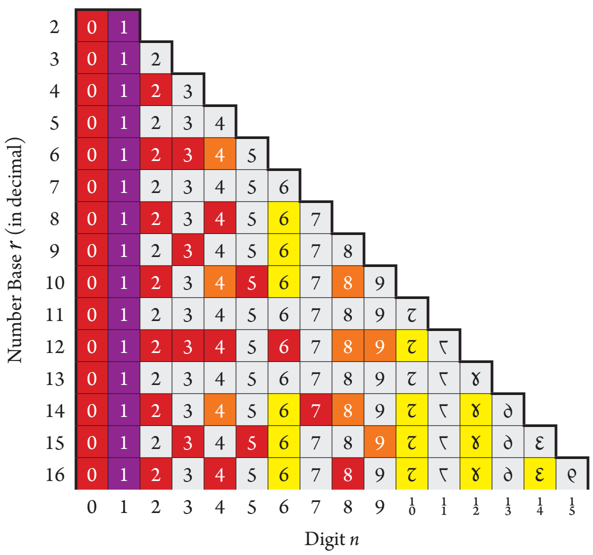


Figure 1.7. A digit map of bases $2 \le r \le 16$.

There are five kinds of digits: units, divisors, and semidivisors are regular digits, and semitotatives and totatives are non-regular.

Now that we know that there are five kinds of digits, let's examine three common applications for number bases.

Expansion of Fractions.

The decimal expansion of fractions that came into use in the middle ages revolutionized computation. The integral and fractional part of numbers could be computed using the same addition and multiplication algorithms. Since we are discussing number bases, let's refer to the expression of fractions using the digits of a number base "digital expansion".

We've already seen that the digital expansion of reciprocals of the divisors *d* of a number base terminate after a single place filled with the divisor complement d'. This is the simplest type of fraction (except perhaps the digital expansion of the fraction $1/1 = 1$.)

Regular numbers are by definition those that have terminating digital expansions. Thus, semidivisors also have terminating digital expansions [40]. The decimal expansions of $1/4 = .25$ and of $1/8 = .125$. The dozenal expansions of $1/8 = ;16$ and $1/9 = ;14$. This works for reciprocals of regular numbers greater than the base. The decimal expansions of $1/625 = .0016$ and $1/32768 = .000030517578125$, the latter, though regular, not very convenient.

The digital expansions of fractions with totatives or numbers coprime to the base in the denominator are purely recurrent[41]. The decimal expansion of $1/7 = .142857\ldots$ and the dozen expansion of the same fraction is ;186X35…. The digital expansions of semitotatives and any product of at least one prime divisor and at least one prime totative will have a brief non-recurrent "preamble" followed by a recurrent, infinite portion (i.e., these are "mixed recurrent" [41]). The decimal expansions of $1/6 = .1⁝666\ldots$ and $1/14 = .0⁝714285\ldots$. The dozenal expansion of $1/X = ;1⁝2497\ldots$.

Generally, divisors are the most preferable denominators because their digital expansions are briefest. Any regular number

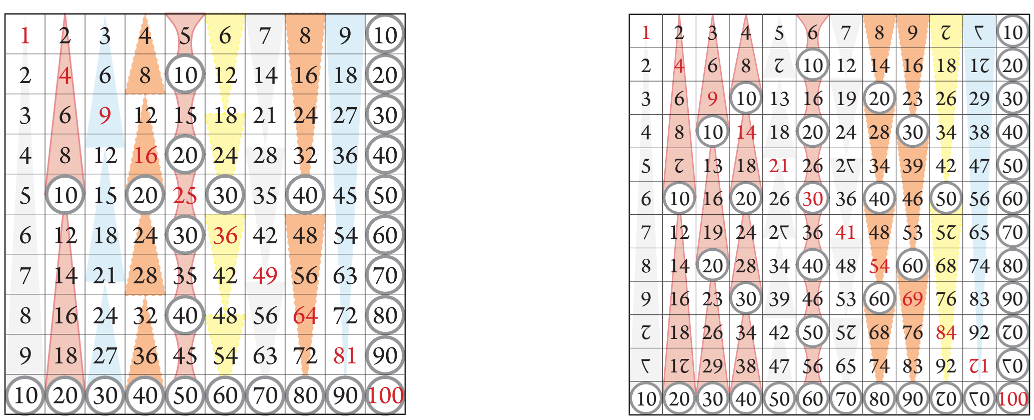


Figure 1.8. Patterns in the decimal multiplication table (left) and the duodecimal multiplication table (right). Patterns of increasing digits are indicated by a tone that widens according to the size of the least significant digit. Red patterns pertain to divisors, orange to semidivisors, which along with divisors comprise a base's regular numbers. Exact multiples of the number base are circled.

will at least have a terminating expansion, though some regular numbers can have very long expansions. The non-regular numbers, including totatives and semitotatives, are the least preferable denominators because they have recurrent digital expansions. A number base that maximizes divisors, semidivisors, and regular numbers in general but minimizes totatives and semitotatives seems to be preferable.

Multiplication Tables.

The multiplication table is an essential learning aid for arithmetic in any number base. We've already seen that divisors have brief patterns, that decimal has two non-trivial divisors {2, 5} and dozenal has four {2, 3, 4, 6}. The totatives of a number base *r* generally feature patterns of end digits in the product line have a maximally-long cycle. The set of end digits of the products of 7 in the decimal table is

{0, 7, 4, 1, 8, 5, 2, 9, 6, 3}

and in dozenal it is similarly jumbled:

{0, 7, 2, 9, 4, Ɛ, 6, 1, 8, 3, Χ, 5}

The neutral digits have cycles that are not maximum and may even be brief, but do not "land" on every multiple of the number base *r* like those of divisors. The end digits of the decimal semidivisor 4 are {0, 4, 8, 2, 6} and the decimal semitotative 6 are {0, 6, 2, 8, 4}. The end digits of the dozenal semidivisor 8 are {0, 8, 4} and the dozenal semitotative Χ are {0, Χ, 8, 6, 4, 2}.

Divisors tend to have the most easily memorized and recalled product lines. The product lines of totatives would seem to be less easily memorized and recalled, especially if the number base is large. Neutral digits can have product lines that are between those of divisors and totatives to memorize and recall. Again, a number base that maximizes divisors while minimizing totatives seems to be preferable. (See Figure 1.8. and Appendix B)

Intuitive Divisibility Tests.

An intuitive divisibility test is a quick and simple method of determining whether an integer is divisible by another. This is a useful application because one can simplify fractions or quickly determine, for instance, whether one can set a group of three colors of tile in a wall 57 tiles wide, and come out with unbroken groups. The intuitive divisibility test is a kind of shortcut. See Appendix C for a map of intuitive divisibility tests for bases $2 \le r \le 60$.

The first class of divisibility rules pertain to regular numbers. Let's call the class the "regular divisibility rules." To determine whether an arbitrary integer *x* is divisible by a regular digit, we examine a finite set of least significant digits [42]. Divisors require us to examine only the very least significant digit, as we've already seen. Semidivisors and regular numbers in general will require us to examine more than one rightmost digit. Example: decimal 744 is divisible by the semidivisor 4 because 744 ends in "44", which is a multiple of 4. The decimal regular rule pertaining to 8 is not as easy to implement. Decimal 1568 is divisible by 8 because the last three digits are divisible by 8. I don't know about you, but I haven't memorized the 125 combinations of three digits that are divisible by 8! Dozenal semidivisors and non-divisor regular digits have similar tests. Dozenal 2468; is divisible by 8 because it ends in "68;", which is a multiple of 8. Dozenal 2468; is not divisible by 14; (one dozen four, which is sixteen) because "468" is not one of the ten dozen eight (decimal 108) combinations of three-digit dozenal numbers that are divisible by 14;.

Let's recognize a special type of totative, because this kind of totative is key to a major pair of intuitive divisibility tests. Let the integers alpha, $\alpha = (r + 1)$, the "neighbor upstairs," and omega, $\omega = (r - 1)$, the "neighbor downstairs." The decimal omega is 9 and alpha is 11. The duodecimal omega is eleven and the alpha is one dozen one.

An integer *x* is divisible by alpha if the difference of the sum of the digits in even places and the sum of the digits in odd places is divisible by alpha [43]. In decimal, 1342 is divisible by 11 because $(1 + 4) - (3 + 2) = 0$, and zero is a multiple of 11. This is the "alternating sum" or "alpha rule".

An integer *x* is divisible by omega if the sum of the digits in all the places of the integer *x* is divisible by omega [44]. Decimal 729 is divisible by 9 because $(7 + 2 + 9) = 18$, and 18 is clearly divisible by 9. Dozenal 2Ɛ9; (decimal 429) is divisible by Ɛ because (2 + Ɛ + 9) = 1Χ;, which is clearly a multiple of Ɛ.

The last intuitive divisibility test is called the "compound test". Decimal has an intuitive rule for the digit 6: if an integer *x* is even and divisible by 3, then it is divisible by 6. Duodecimal does have such rules but they pertain to numbers larger than twelve, since there are no neighbor-related rules for digits less than eleven.

The decimal omega is composite, which is fortuitous. This is because the "benefits" of the neighbor-related numbers alpha and omega are "inherited" by their factors. Additionally fortunate is the fact that all factors of a number coprime to the base are also coprime, helping to ameliorate the difficulty presented by numbers coprime to the base. We can call a totative that does not benefit from neighbor-relatedness an "opaque totative", because these do not have intuitive rules that facilitate their use as tools of computation.

Since 3 is a factor of 9, the omega rule functions for 3 in base ten. This is why we can use the digit sum rule to determine if an integer is divisible by three in base ten.

Because the duodecimal neighbors are both prime, they do not "share" the neighbor-related divisibility rules with smaller numbers. This is unfortunate, because this leaves us with no intuitive divisibility test for the number 5 in base twelve.

Neighbor-relatedness also mildly affects totatives in the multiplication table. More significantly, omega-relatedness minimizes the cycle of recurrent fractions, while alpha-related totatives have a brief, 2-digit recurrent cycle. In base ten, ⅓ = .333… and ⅑ = .111…, while 1⁄11 = .090909….

Hexadecimal has ω = fifteen. Thus, not only does the digit sum rule work for 3, but also for 5. Additionally, ⅓ = .555… and ⅕ = .333…. Base 26 has $\omega = 25$ and

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Figure 1.10 a. Decimal arrangement of coprime primes with end digits that are decimal totatives {1, 3, 7, 9}. Primes in red are decimal divisors.

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
11	12	13	14	15	16	17	18	19	1χ	1Ɛ	20
21	22	23	24	25	26	27	28	29	2χ	2Ɛ	30
31	32	33	34	35	36	37	38	39	3χ	3Ɛ	40
41	42	43	44	45	46	47	48	49	4χ	4Ɛ	50
51	52	53	54	55	56	57	58	59	5χ	5Ɛ	60
61	62	63	64	65	66	67	68	69	6χ	6Ɛ	70
71	72	73	74	75	76	77	78	79	7χ	7Ɛ	80
81	82	83	84	85	86	87	88	89	8χ	8Ɛ	90
91	92	93	94	95	96	97	98	99	9χ	9Ɛ	χ0

Figure 1.10 b. Duodecimal arrangement of coprime primes with end digits that are duodecimal totatives {1, 5, 7, Ɛ}. Primes in red are dozenal divisors.

1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10
11	12	13	14	15	16	17	18	19	1a	1b	1c	1d	1e	1f	20
21	22	23	24	25	26	27	28	29	2a	2b	2c	2d	2e	2f	30
31	32	33	34	35	36	37	38	39	3a	3b	3c	3d	3e	3f	40
41	42	43	44	45	46	47	48	49	4a	4b	4c	4d	4e	4f	50
51	52	53	54	55	56	57	58	59	5a	5b	5c	5d	5e	5f	60
61	62	63	64	65	66	67	68	69	6a	6b	6c	6d	6e	6f	70
71	72	73	74	75	76	77	78	79	7a	7b	7c	7d	7e	7f	80

Figure 1.10 c. Hexadecimal arrangement of coprime primes with end digits that are hexadecimal totatives (odd digits).

a = 27, thus enjoys neighbor related benefits for the digits {3, 5, 9, 25}.

We can show neighbor-related totatives in digit maps.

0	1	2	3	4	5	6	7	8	9

Figure 1.9. A digit map of base r = 10, *showing omega totatives in light blue.*

Bases that have plenty of divisors have plenty of easy divisibility tests. Semidivisors and non-divisor regular numbers often have simple tests, but some of these can be impractical. A number base with a great deal of "opaque totatives" presents resistance to human computation. One that has one or more composite "neighbors" will have some totatives with neighbor-related divisibility tests as well as brief recurrent digital expansions. This helps ameliorate the difficulty totatives present to human intuition in computation.

Limiting Totative Resistance

The following is even more technical than some of this information, however it does illustrate that duodecimal minimizes resistance presented by totatives.

In addition to the generally resistive effect of totatives in the three common applications described above, they have another major application. The totatives of a number base "arrange" prime numbers. A number base r will express prime numbers q that are coprime to r with a number having a least significant digit that is a totative. Decimal does not have an unpleasant number of totatives, so the less-dense distribution of primes presented by dozenal is not as keenly noticed. However, hexadecimal and octal can harbor primes on any odd digit, which is far less convenient. See Figure 1.10. (For an excellent interactive illustration of the arrangement of primes in various number bases, visit http://demonstrations.wolfram.com/DistributionOfPrimes/.)

The Euler totient function $\phi(r)$ counts totatives of r [24, 25]. The Euler totient function $\phi(p)$ for a prime p is:

(1.1) $$\phi(p) = p - 1, \text{ [26]}$$

The Euler totient function [27, 28] for all composite numbers r is

(1.2) $$\phi(r) = r \cdot \prod_{i=1}^{k} (1 - 1/p_i) = r(1 - 1/p_1)(1 - 1/p_2)\ldots(1 - 1/p_k).$$

From formula (1.2) we can see that each distinct prime divisor p_i contributes one factor $(1 - 1/p_i)$. The factor is reliant on the magnitude of p_i but the exponent of p_i is immaterial in each factor.

If we consider the totient ratio

(1.3) $$\phi(r)/r = \prod_{i=1}^{k} (1 - 1/p_i)$$

the base r and the magnitude of its digit range are scaled to 1, and we can observe the effects of distinct prime factors on the proportion of totatives in base r. (See Figure 1.11.) We only need to consider squarefree versions ρ of bases r such that

(1.4) $$r = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k} \longrightarrow \rho = p_1 p_2 \ldots p_k$$

e.g., $12 = 2^2 \cdot 3 \longrightarrow 6 = 2 \cdot 3$ and
$360 = 2^3 \cdot 3^2 \cdot 5 \longrightarrow 30 = 2 \cdot 3 \cdot 5$.

Dozenal has the distinct prime divisors {2, 3}, scoring a totient ratio of ⅓ while decimal has {2, 5} with a totient ratio of ⅖. Hexadecimal has one prime divisor {2} and a totient ratio of ½. Dozenal thus has less of "the bad stuff". (See Figure 1.11; twelve appears at "2·3" while ten appears at "2·5".)

These applications aren't the only important ways a number base is used as a technology, but they are key ways. A less-important application is the examination of end-digits of perfect squares. In this case, duodecimal perfect squares end in one of {0, 1, 4, 9} while decimal perfect squares end in one of {0, 1, 4, 5, 6, 9}. Most of the merits of a number base can be understood at a glance through examination of the number-theoretical relationships between base r and its digits n.

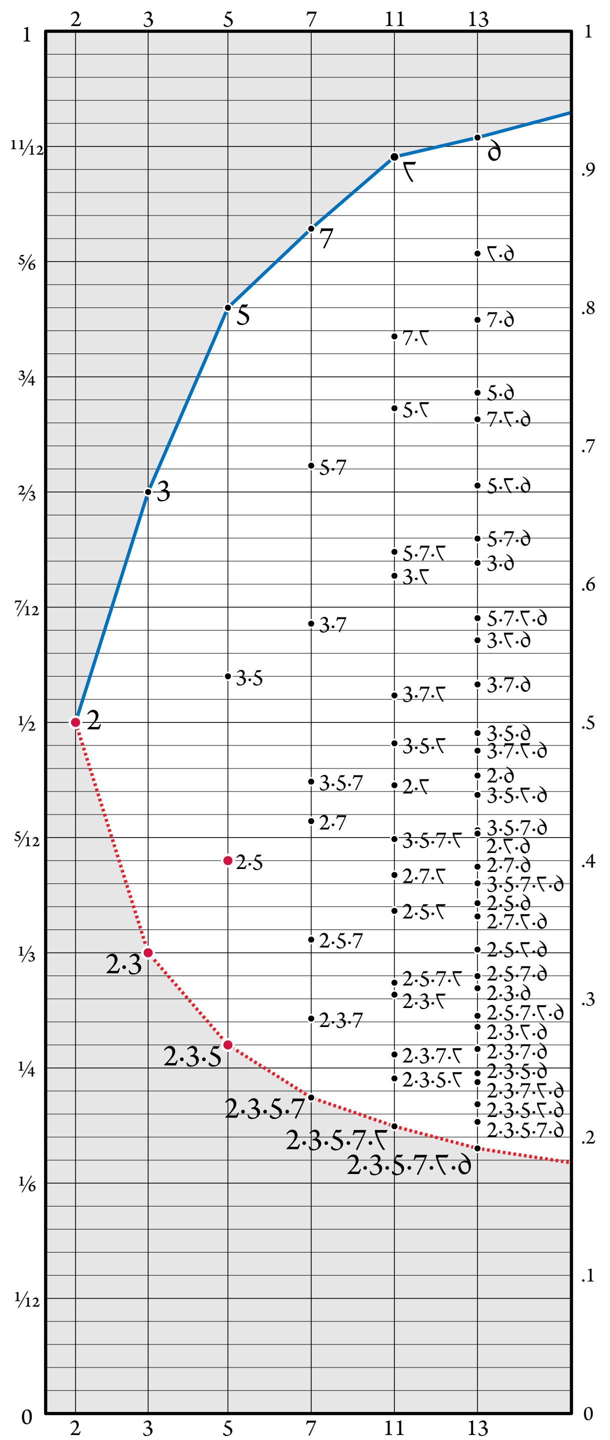


Figure 1.11. A plot with φ(r)/r on the vertical axis versus the maximum distinct prime divisor p_{max} on the horizontal axis. (The horizontal axis is not to scale.) The p_{max}-smooth numbers lie along a vertical line at each value of p_{max}. The boundary of minimum values of φ(r)/r defined by primorials is indicated by a broken red line. The boundary of maximum values of φ(r)/r defined by primes is shown in blue. All other composite numbers r that have p_{max} as the maximum distinct prime divisor inhabit the region between the boundaries. *See page 19;* *for more scope, running to the prime 61.*

Enriching Regular Digit Density

While duodecimal acts to reduce the totient ratio, it also enriches the density of regular digits that tend to facilitate human intuitive manipulation of numbers.

Let's start off by examining the prime decomposition of 10 = 2 · 5. Ten has two distinct prime factors, enabling it to have regular numbers that are not powers of prime factors. Decimal has more regular numbers between 1 and 256 inclusive than does hexadecimal: {1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100, 125, 128, 160, 200, 250, 256} = 21 [58] versus {1, 2, 4, 8, 16, 32, 64, 128, 256}, only nine. Keep in mind sixteen has 5 divisors while ten has 4. This means that decimal has more terminating unit fractions than hexadecimal. This is still fewer than dozenal, which has {1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 81, 96, 108, 128, 144, 162, 192, 216, 243, 256} = 28 [59].

We can examine this in two ways. Like Figure 1.10 illustrates the arrangement of primes *q* that are coprime to *r*, we can produce Figure 1.12 that shows regular digits *g* below a threshold. Let's use a threshold around 120 to 128 so we can "be fair".

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Figure 1.12 a. Decimal regular numbers 1 ≤ g < 120.

1	2	3	4	5	6	7	8	9	X	Ɛ	10
11	12	13	14	15	16	17	18	19	1X	1Ɛ	20
21	22	23	24	25	26	27	28	29	2X	2Ɛ	30
31	32	33	34	35	36	37	38	39	3X	3Ɛ	40
41	42	43	44	45	46	47	48	49	4X	4Ɛ	50
51	52	53	54	55	56	57	58	59	5X	5Ɛ	60
61	62	63	64	65	66	67	68	69	6X	6Ɛ	70
71	72	73	74	75	76	77	78	79	7X	7Ɛ	80
81	82	83	84	85	86	87	88	89	8X	8Ɛ	90
91	92	93	94	95	96	97	98	99	9X	9Ɛ	X0

Figure 1.12 b. Duodecimal regular numbers 1 ≤ g < 120.

1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10
11	12	13	14	15	16	17	18	19	1a	1b	1c	1d	1e	1f	20
21	22	23	24	25	26	27	28	29	2a	2b	2c	2d	2e	2f	30
31	32	33	34	35	36	37	38	39	3a	3b	3c	3d	3e	3f	40
41	42	43	44	45	46	47	48	49	4a	4b	4c	4d	4e	4f	50
51	52	53	54	55	56	57	58	59	5a	5b	5c	5d	5e	5f	60
61	62	63	64	65	66	67	68	69	6a	6b	6c	6d	6e	6f	70
71	72	73	74	75	76	77	78	79	7a	7b	7c	7d	7e	7f	80

Figure 1.12 c. Hexadecimal regular numbers 1 ≤ g < 128.

1									
2	4								
3	6	9							
4	8	12	16						
5	10	15	20	25					
6	12	18	24	30	36				
7	14	21	28	35	42	49			
8	16	24	32	40	48	56	64		
9	18	27	36	45	54	63	72	81	
10	20	30	40	50	60	70	80	90	100

Figure 1.13 a. Relationship of products in the decimal multiplication table with r = *10.*

1											
2	4										
3	6	9									
4	8	10	14								
5	Χ	13	18	21							
6	10	16	20	26	30						
7	12	19	24	2Ɛ	36	41					
8	14	20	28	34	40	48	54				
9	16	23	30	39	46	53	60	69			
Χ	18	26	34	42	50	5Χ	68	76	84		
Ɛ	1Χ	29	38	47	56	65	74	83	92	Χ1	
10	20	30	40	50	60	70	80	90	Χ0	Ɛ0	100

Figure 1.13 b. Relationship of products in the duodecimal multiplication table with r = *12.*

1															
2	4														
3	6	9													
4	8	c	10												
5	a	f	14	19											
6	c	12	18	1e	24										
7	e	15	1c	23	2a	31									
8	10	18	20	28	30	38	40								
9	12	1b	24	2d	36	3f	48	51							
a	14	1e	28	32	3c	46	50	5a	64						
b	16	21	2c	37	42	4d	58	63	6e	79					
c	18	24	30	3c	48	54	60	6c	78	84	90				
d	1a	27	34	41	4e	5b	68	75	82	8f	9c	a9			
e	1c	2a	38	46	54	62	70	7e	8c	9a	a8	b6	c4		
f	1e	2d	3c	4b	5a	69	78	87	96	a5	b4	c3	d2	e1	
10	20	30	40	50	60	70	80	90	a0	b0	c0	d0	e0	f0	100

Figure 1.13 c. Relationship of products in the hexadecimal multiplication table with r = *16.*

The unit (digit 1) appears in purple, divisors in red, and regular non-divisors *g* in orange. It's clear that hexadecimal, a power of a prime *p*, is bereft of regular digits. Decimal and dozenal appear to have quite a few options for terminating digital fractions.

The second way to examine regular digits of base *r* is to look at the unique product combinations in the multiplication table of base *r*. We can color-code the products according to their relationship with *r*. Figure 1.13 uses the same color scheme as Figure 1.12, adding the non-regular numbers. The hexadecimal table appears void of regular numbers, while the decimal and duodecimal tables are rich with them.

The practical application of an enriched set of regular digits is that the dozenal analog of percents (%), "per gross" ($^{P}/_{G}$), resonate with more useful terminating fractions than any number base between octal and hexadecimal. Figure 1.14 shows how common proportions appear in the analogs to decimal percent across decimal, duodecimal, and hexadecimal. This dovetails with the keen handling of fractions, so consider this a resource for the last question as well.

The keen set of regular digits goes beyond analogs to percentages. The common 3-significant digit representation of figures can be awkward in decimal. Suppose you derive a result from a computation like 3⅓ units, or 6⅞ units. The decimal rounding of these figures to 3.33 or 6.88 units introduces needless error to the calculation. The duodecimal rounding to three digits would preserve the fractional part of these figures: 3;40 and 6;Χ6. Because dozenal has a keen relationship with its aliquot parts, it serves even to aid our approximations.

2. *Do you think the world should/will change to the dozenal system?*

A. [The following represents the author's opinion and is not necessarily the opinion of the Dozenal Society of America].

Let's start off by recognizing that a system of numeration is a *technology*. It is a tool that facilitates human understanding and manipulation of real quantities. Mankind builds upon its technologies to produce further inventions. Decimal is deeply entrained in our current civilization. In ancient times, only merchants and scribes recorded numbers. With the printing press, numbers and language could be printed en masse. The industrial revolution began to incorporate mechanical manifestations of the decimal number base. Greater education brought literacy and numeracy to the public. Broadcasting and literacy truly entrained the decimal system; the information age has intensified decimal exponentially. Not only is our mechanical hardware decimal, but also our software. Underlying the user interface of our software, binary is the basis of our computer technology, with hexadecimal serving to help programmers interpret binary data and reduce transcription errors. What had served as a programmer's tool has become accessible to graphic designers as the means to specify screen color. Decimal is highly interwoven into our civilization; the time to have changed bases was perhaps before the industrial revolution. Hexadecimal is an auxiliary technology that serves a niche purpose.

There are two questions here: Should the world be dozenal, and could the world be dozenal.

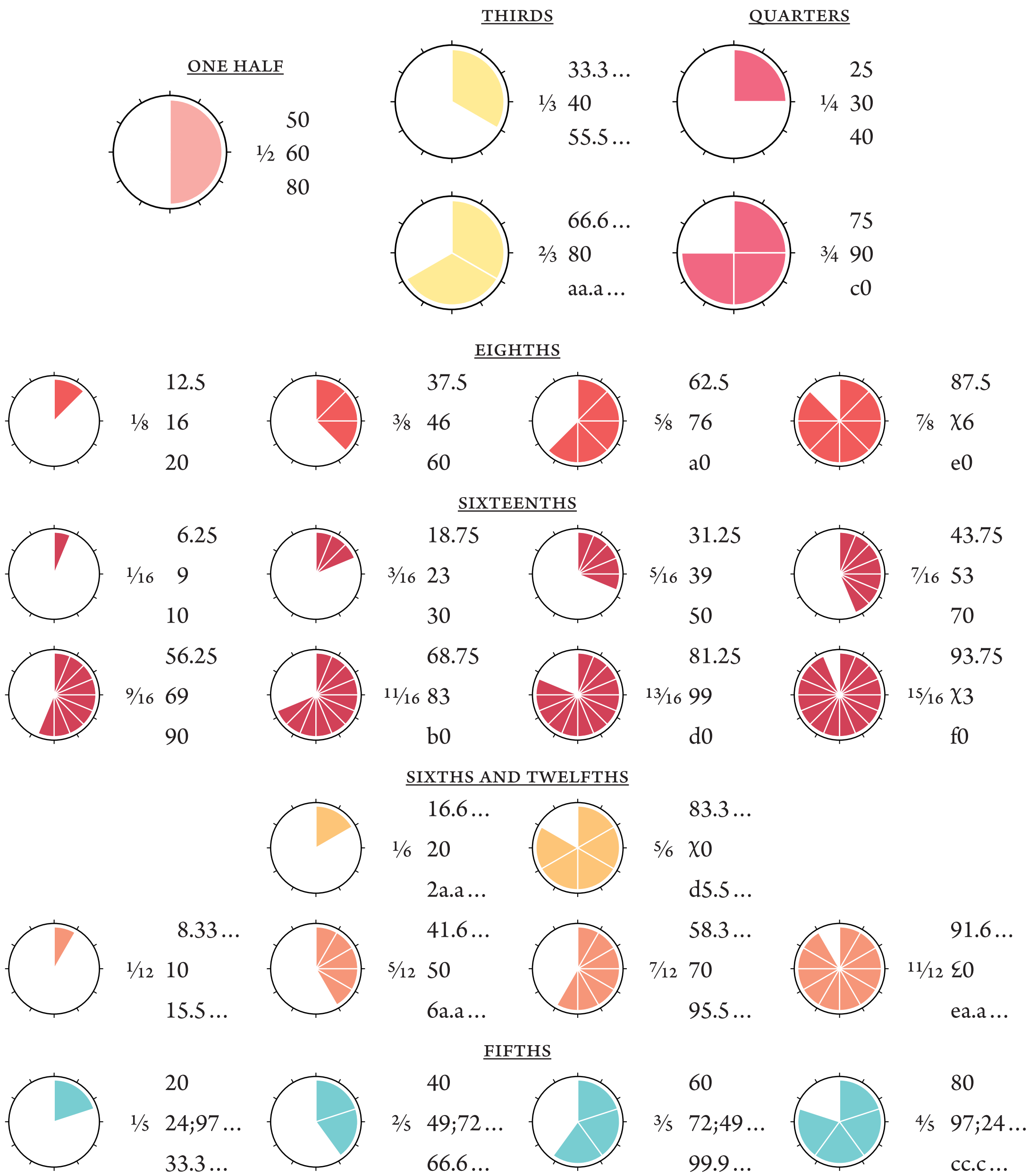


Figure 1.14. Common fractions expressed in analogs to percent. The top figure is decimal, middle dozenal, and bottom hexadecimal for each fraction.

First, the world would run more efficiently if it were based on dozens rather than tens. Less time would be necessary in educating the young regarding arithmetic and mathematics would thereby be simpler. The world should have based itself on duodecimal rather than decimal. It seems the last time this change might have been feasible was in the 18th century, perhaps when what has become the Systeme Internationale (metric system) was conceived. Even then, a change may have been too late. The change would have had to affect the British Empire and continental Europe before their colonies fragmented and the revolutions of the 18th century took place. Then the American currency would not have decimalized, and the metric system in place would instead be dozenal.

The world will not likely "convert" to dozenal. Instead, duodecimal numeration may continue to find use as an auxiliary number base. Duodecimal may find a future application, much like hexadecimal has in the configuration of bits into parcels that facilitate manipulation by human programmers. Duodecimal will likely serve along with a great number of auxiliary bases like 16, 24, 60, and 360 well into the future.

In many ways, we are today a duodecimal civilization. Ever ask someone if 48 or 72 were "round" numbers? Ever ask someone if the number 12 or 24 were somehow weird? The fact is that the world configures items by the dozen because it is a convenient number. Merchants have been purveying ware by dozens and grosses for centuries. Glancing in a catalog printed only yesterday, we can see pens and crayons sold in dozens. Sometimes tradition goes beyond simply doing things because that's how it's been done; sometimes our forefathers had a keen system in place; their wisdom made them successful. Some traditional systems are simply smart solutions that have functioned well for centuries because they are optimum.

3. *What things (if any) are being done to educate the world about the duodecimal system?*

A. The Dozenal Society of America was founded in 1944 as a "voluntary nonprofit educational corporation, organized for the conduct of research and education of the public in the use of base twelve in calculations, mathematics, weights and measures, and other branches of pure and applied science." In the past, in addition to the work of notable mathematicians Pascal and Laplace, other luminaries have disseminated insight regarding duodecimal numeration [45]. In England, Sir Issac Pitman wrote about the duodecimal system in the 9 February 1856 issue of his *Phonetic Journal*. Pitman, according to Wikipedia, "invented the most widely used system of shorthand" [46]. F. Emerson Andrews, a Founder of the Duodecimal Society of America, wrote "An Excursion in Numbers" in the *Atlantic Monthly* in October 1934, introducing America to duodecimal numbers [47, 48]. In 1955, the French author Jean Essig wrote *Douze notre dix futur* (French: *Twelve, the Future Ten*) [49], and in 1959, the Duodecimal Society of Great Britain was founded [50]. The American Broadcasting Company produced a series of video shorts called "Schoolhouse Rock!" broadcast during Saturday morning children's programming in the 1970s and 1980s. One of the segments, "Little Twelvetoes", was produced and first aired 9 March 1973 [51, 52]. This interstitial program used the numerals printed in the *Duodecimal Bulletin* to introduce duodecimal multiplication to children. This brief introduction served to explain the products of twelve in the decimal multiplication table.

On 18 July 1995, Prof. Gene Zirkel was interviewed on National Public Radio's "All Things Considered" on the merits of duodecimal numeration [53].

In short, the Dozenal Societies of America and Great Britain have used print, web, and interview to help educate the public on the merits of duodecimal numeration on a volunteer basis. From time to time, the wider society picks up on the publications and communications of the DSA and DSGB, resulting in broadcasts like that of the *Atlantic Monthly* in the 1930s, the Saturday morning cartoon "Little Twelvetoes" in rotation across a couple decades on the ABC television network, and the NPR interview of 1995.

4. *If certain places decided they did want to change to the dozenal system, would they just change the base in counting and mathematics or also change the measurements they use? (Length, weight, time, etc.)*

[The following represents the author's opinion and is not necessarily the opinion of the Dozenal Society of America].

A. It's difficult to prognosticate how a society would adapt to the alteration of a technology like the number base, which has by now become fundamental to everyday life. Let's suppose that tomorrow, our community began using duodecimal arithmetic and numerals, that the change would be instant, without a learning curve or transition, and let's assume that only the numerals, number words, and arithmetic changed. We would then have a system of measure, whether that be the decimal metric system or the US Customary / Imperial system of measure, or any other extant system, based in part or total on a non-dozenal number base. In most cases this would be awkward. There would probably be a move to produce a system of weights and measure that would match the number base, a dozenal metric system. This system might resemble Tom Pendlebury's TGM [54] or Takashi Suga's Universal Unit System [55]. Additionally, currency would likely be brought in line with the number base. Instead of a dollar or euro of 100 cents, a pound of 100 pence, we'd likely see a unit of 144 parts soon develop. Conversion between decimal and dozenal would be important if a single community changed over, since others would be using decimal, so along with language differences, there would be a numeration difference requiring translation. If all humanity changed to dozenal, we would still require "translation" of figures in historical documents, as all human records before the change would be decimal (or in other bases).

5. *Assuming that the people agree that the duodecimal system is better than the decimal system, how could people go about changing from one system to the other?*

[The following represents the author's opinion and is not necessarily the opinion of the Dozenal Society of America].

A. There are people who accept that duodecimal is optimum. These people may use duodecimal arithmetic in their work and their everyday life. I'll use myself as an example. I am an American architect who uses the US Customary system of measure, as metric does not appear to have a chance of usurping the use of US Customary in my lifetime for a variety of sociopolitical reasons, feet and inches are the way lengths are reckoned on most building projects in the USA. When I take field measurements, I record them using duodecimal figures. Let's suppose we need to measure a room that is 13′-1½″ (approximately 4 meters) wide. I would write 111;6 inches or 11;16 feet. If I had a series of dimensions, I could total them in duodecimal, then divide them by 3 to arrive at window spacing or distance between beams on center, etc., just as one can do in the decimal metric system. However, duodecimal feet and inches are superior in that a precise third and quarter can be represented simply in duodecimal, while in decimal metric, thirds tend to be avoided or rounded, or a module divisible by twelve, such as 1200 mm gets used.

Even if one doesn't use duodecimal arithmetic professionally, one can use it in everyday life. For example, I am a swimmer; I can count my laps in duodecimal. Today I swam four dozen laps in three dozen four minutes. That's an average of 0;Ϫ minutes (50 seconds) per lap. Oftentimes my wife is cooking and wants to divide the recipe. Usually the divisions happen to be in thirds, quarters, sixths, or eighths for one reason or another. These divisions are simpler in duodecimal than in decimal.

Having an application for duodecimal numeration will certainly augment its use. The computer science community uses hexadecimal as a tool in interpreting bits. This has telegraphed into the hexadecimal specification of screen colors by graphic designers, much-removed from the original purpose of the computer scientist. Without a purpose, there really isn't a practical need to convert to duodecimal.

The best thing that a "dozenalist" can do to further the use of duodecimal is to apply it in his or her daily arithmetic.

6. *What are the main disadvantages to the duodecimal system in real life situations? (Comparing to the system we use today)*

A. [The following represents the author's opinion and is not necessarily the opinion of the Dozenal Society of America].

The Achilles' heel for duodecimal is the fact that it possesses two "opaque" totatives, 5 and 7. Decimal possesses only one "opaque" totative. Five is an important prime number, not as important as two or even three, but certainly not unimportant. Fivefold symmetry isn't as common in geometry and biology as threefold, but it is still encountered. Pentagons can produce regular three-dimensional objects, thus is important not only in geometry but chemistry and physics. This said, name all the buildings you know that feature pentagonal symmetry. The fact you can think of only one shows that five is not quite as important as two, three or their products in our society. In the decimal system, 3 is related to the "omega totative" 9. This means we can use the digit-sum divisibility test to determine if a decimal integer is a multiple of three. We can't do the same for 5 in base twelve—there is no intuitive divisibility test. One fifth has a maximally recurrent duodecimal expansion in duodecimal: 0;24972497…, just about as messy as one seventh in either decimal or duodecimal.

The fact that five is opaque in duodecimal maroons the duodecimal semitotative ten, leaving it without an intuitive divisibility test. It would be important for a society using duodecimal to memorize the less-intuitive divisibility test for 5 so that people could have divisibility tests for its multiples, which are rather common, though not quite as common as multiples of 2, 3, or combinations thereof.

The dozenal omega digit, eleven, is not very helpful. We seldom need to use elevenfold symmetry or determine whether a number is a multiple of 11 in daily life, commerce, and industry. Decimal is equipped with the ability to test for eleven-ness through the alternating-sum divisibility test anyway. The decimal omega digit, 9, has a factor 3 that inherits its digit-sum divisibility test. Since eleven is prime, that grace does not befall duodecimal. The neighbor related digit divisibility tests in duodecimal, applying to the primes 11 and 13, simply aren't very helpful.

All this seems to imply that a human society using duodecimal will tend to be more "allergic" to factors of five, fifths, and their products than the current decimal civilization is to thirds, sixths, twelfths, etc. This is only amplified by the fact that dozenal works so well with halves, thirds, quarters, sixths, eighths, etc. I think we would nearly never encounter anything divided by five or ten in a dozenal civilization, just as sevenfold-ness is rare in decimal civilization. In this way, decimal perhaps makes us better arithmeticians, because we are forced to use transparent totatives (3 and 9). In fact, I think in a decimal culture we are very accustomed to multiples of three, six, and twelve, even though our system doesn't "resolve" them as well as duodecimal would.

The fact that twelve is "allergic" to five in a way that renders five as "weird" as seven, to a more extreme degree than ten is alienated from three, bugs me.

This is not to say that there *aren't* dozenal divisibility tests for five. Clearly there are, and are produceable using modular maths. Here are two I geeked out using the relationship of $12n$ modulo 5 [56].

Duodecimal divisibility tests for five using the least significant digit (all arithmetic is duodecimal):

We must solve

$$10;n = 1 \pmod 5,\ n = 3,$$

thus take the last digit, multiply it by 3, then add the product to the remaining digits. Example, 89; is divisible by 5 because

$$8 + 3(9) = 8 + 23; = 2\text{↋}; = 5(7)$$

[89; is decimal 105]. The integer 286; is divisible by 5 since

$$28; + 3(6) = 28; + 16; = 42; = 5(\text{↊})$$

[286; is decimal 390]. The decimal power 49,↊54; is divisible by five because

$$49\text{↊}5; + 3(4) = 49\text{↋}5,$$

which is divisible by five because

$$49\text{↋}; + 3(5) = 4\text{↋}1;,$$

which is divisible by five because

$$49; + 3(\text{↋}) = 49; + 29; = 76; = 3(26;) = (2 \cdot 3^2 \cdot 5)$$

[49,↊54; is decimal 100,000.]

Duodecimal divisibility tests for five using two least significant digits (again, all arithmetic is duodecimal):

We must solve

$$100;n = -1 \pmod 5,\ n = 1,$$

thus take the last two digits and subtract them from the rest. Example, 286; is divisible by five because

$$86; - 2 = 84;, = \text{↊}^2 = 5(18;).$$

The integer 1977; is divisible by 5 since

$$77; - 19; = 5\text{↊}; = 5(12;)$$

[1977; is decimal 3115]. The decimal power 49,↊54;, is divisible by five because

$$49↊; - 54; = 446;,$$

which is divisible by five because

$$46; - 4 = 42; = 5(↊).$$

Given some tests for five, we have a compound method for testing divisibility by ten in base twelve. If a number is even and divisible by five, then it is divisible by ten. Thus the numbers 286; and 49,↊54; are divisible by ten since they are even and divisible by five. The numbers 89; and 1977; are not divisible by ten, since they are odd.

Rules like these wouldn't seem "intuitive" because the average guy watching TV would never associate tearing off the last digit and multiplying it by three before subtracting the product from the rest of the digits to see if something is divisible by five. The average person is not likely to know modular maths as well.

7. *The decimal system is the dominant system today. What good properties make the decimal system superior to other systems?*

A. I wrote a post at the DozensOnline forum called "Dare I admit good things about decimal?", which I am gleaning to answer this question completely [57].

I am very nearly certain that decimal is the second- or third-best number base for general human computation. Let's start off by examining the prime decomposition of $10 = 2 \cdot 5$. Ten has two distinct prime factors, enabling it to have regular numbers that are not powers of prime factors. Decimal thus has the ability to arrange primes q coprime to 10 under just 4 totatives {1, 3, 7, 9} (see Figure 1.10 a). Additionally, decimal has a relatively high density of regular numbers (see Figures 1.12 a and 1.13 a). Let's recap the decimal digit map:

0	1	2	3	4	5	6	7	8	9

Figure 7.1. The decimal digit map, with the unit in purple, divisors in red, semidivisors in orange, semitotative in yellow, omega related totatives in light blue, and opaque totatives in light gray.

0	1	2	3	4	5	6	7	8	9	↊	↋

Figure 7.2. The duodecimal digit map.

0	1	2	3	4	5	6	7	8	9	a	b	c	d	e	f

Figure 7.3. The hexadecimal digit map.

This article is a sort of rejoicing for the fact we didn't end up with a worse base. Oftentimes, in our effort to illustrate the advantages of dozenal, we tend to bash decimal or downplay its attributes. Most people likely never examine the number base in play; base ten is as taken for granted as breathing the air around us. This thread is not an apology for decimal. We'll see that some of the graces decimal enjoys are not necessary given the greater divisibility of other number bases. This is simply an acknowledgement that we might have ended up with a worse base of computation. As one who accepts that dozenal is the optimum number base for general human computation, I believe it is a testament to the strength of twelve as a base to objectively examine and acknowledge the beneficial qualities and arrangement of decimal. Let's take a closer look at our native number base.

Decimal sports four divisor digits {0, 1, 2, 5}, and four digits out of phase with ten, the totatives {1, 3, 7, 9}. The former are aids to computation while the latter tend to present resistance to human intuition. Of the four primes {2, 3, 5, 7} less than ten, two are represented in decimal prime factorization, the set of prime divisors of ten {2, 5}, and two are left out as prime totatives {3, 7}. A user of decimal thus perceives the permutations of the prime divisors {2, 5} (especially when these numbers are small) as "regular", friendly numbers, with keen, terminating fractions, easy divisibility rules, and simple multiplication fact cycles. Along with the digits 2 and 5, the digits 0 and 1 are easy to work with in decimal, as they are in any base. The digits {4, 8} are semidivisors, friendly and clean, enjoying terminating decimal representations of their reciprocals. The semitotative digit {6}, product of the prime divisor 2 and the prime totative 3, presents a little trouble; its reciprocal has a repeating decimal representation, and it seems to "skip around" in the multiplication table. Six of the ten decimal digits {0, 1, 2, 4, 5, 8} are regular. The other four {3, 6, 7, 9} are either totatives or semitotatives which suffer recurrent decimal fractions, have more difficult patterns in the decimal multiplication table, and cannot use the lesser-significant place values of an integer to determine divisibility in an intuitive way.

Ten enjoys two graces; we would fail to be impartial if we ignore these.

Firstly, if we look at the list of totatives, we see that the last one, let's call $\omega = (r - 1)$ or "omega totative", is 9. This "omega digit" is special in all bases: they are always totatives, their reciprocals always enjoy a single repeating digit (.111…) as a digital fraction, and the "digit sum" divisibility test applies when attempting to detect divisibility of an arbitrary integer x by $(r - 1)$. Most importantly, these keen qualities are inherited by their factors. Nine is the square of three, thus, two decimal totatives {3, 9} enjoy a sort of "phantom divisibility". The fractions ⅓ and ⅑ and their multiples, are simply repeating single-digit mantissas (⅓ = .333… and ⅑ = .111…). As shown before, decimal prime divisors include {2, 5} but miss the interposing prime 3; this omega-related totative status for 3 sort of fills in what's missing regarding three. The fact that the omega totative is the square of three accentuates the "help" decimal gets from its neighbor downstairs (9). Hexadecimal enjoys a similar quality: both 3 and 5 are omega-related prime totatives, yet $9 = 3^2$, remains "unreachable" and relatively opaque to intuitive leverage. To top it off, the decimal semitotative 6 is lent divisibility rules by 2 and 3, so that we have a means of intuitively testing divisibility by 6. So the decimal digits which are multiples of three {3, 6, 9} are somewhat covered by the benefical position of the composite totative 9, the square of the prime 3, missing from the prime decomposition of ten. In decimal, although we do not enjoy direct divisibility by 3, the omega totative inheritances allow us to test for 3, and ease the recurrent digital fractions associated with the thirds and the ninths. If we subtract this list of somewhat ameliorated totatives and semitotatives from the "bad" list {3, 6, 7, 9}, we have only one decimal "opaque" totative: 7. (See Appendix C for a map of intuitive divisibility rules for $2 \le r \le 60$.)

Secondly, decimal enjoys relatively compact and optimal auxiliary bases. An auxiliary base is one which we use to augment or obtain divisibility for reciprocals of numbers outside of a number base's regular numbers, especially for cyclical measurements in a given number base. Notably, we use sexagesimal in the reckoning of the fractions of an hour and minute, and the subdivisions of a degree of arc. We use 360 degrees in the division of the circle. The use of these numbers lends a decimal civilization a "clean" third (i.e. a single-significant-figure representation: ⅓, natively in decimal = .333..., becomes 20 in decimal sexagesimal) and divisions related to reciprocals of multiples of 3, the missing prime. You'll hear "see you in ten minutes" and it sounds nice and round, but we're talking about a sixth of an hour, while it would be a rare event to hear "see you in twelve minutes" or "a fifth of an hour". The latter phrase ought to be common in a fully decimal world, as five is a divisor of ten: if we had fully metric time this awkwardness would be forced upon us. Everyone knows and loves the 45 and 30 degree angles; these are natural, geometrically mandated special angles. The fifth-circle is assigned a less "clean" 72 degrees, but still an integral number of degrees. See the auxiliary base thread for auxiliary bases in other number bases, and you'll see the decimal arrangement is pretty good. The highly factorable dozen as a number base might use a bit of "help" in resolving fifths and tenths; using dozenal sexagesimal (5 on 12) goes too far and overaccentuates the fifths, blunting the natural dozenal resolution of thirds and quarters, even halves. Dozenalists would need to resort to dozenal "500" (decimal 720) for a minimal auxiliary base that guarantees clean halves, thirds, quarters and sixths while resolving fifths. Then again, the dozenal rationale for an auxiliary base is extension of resolution to the next most common prime (five) rather than infill in the case of decimal. It's because decimal features a split prime factorization, where the gap is minimal and the factorization otherwise compact, that 60 and 360 are quite so handy. Pure dozenal cyclical divisions might tend to appear in a dozenal civilization, purely binary divisions perhaps in an octal or hexadecimal civilization, the latter suffering all the more because of it.

In summary, decimal is missing the second most important prime, 3, from the set of its prime factors {2, 5}, which is not quite as efficient as if it were to have incorporated 3 in place of 5 (giving us base 6) or in addition to 5 (yielding base 30). Two convenient avenues provide a means for decimal civilization to "get around" the problem of the missing three. The benefits presented by the decimal omega totative, 9, are inherited by 3 which is a factor of 9. The divisibility test benefits are transmitted to 6 via evenness and the digit sum rule for three. This means decimal users enjoy a sort of "phantom divisibility" for three which enables detection of 3 and 3^2 in decimally-expressed integers, and many of the multiples of three via compounding the regular intuitive divisibility tests. Decimal is well-stocked with intuitive divisibility tests, delivered by its regular digits and its omega totative, covering nine of its ten digits. The second avenue is the effective availability of highly factorable auxiliary bases which resolve the third, maintain the half and the quarter, and only mildly punish the fifth. Decimal civilization tends to accept the blunted fifth in its cyclical measure, favoring clean halves, thirds, and quarters. Thus, decimal civilization is able to work with the first three prime factors satisfactorily, though the omega totative benefits and the highly factorable auxiliary bases. These aids more or less seem to wipe out the decimal blind spot regarding the missing prime factor three, for some applications. Don't get me wrong, thirds are still reciprocals of a decimal totative and behave that way, but they might have been "opaque" to our intuition if our number base weren't congruent to 1 (mod 3).

8. *Today we divide a circle in 360 degrees and in 2π in radians. How would having a system with base 12 benefit the division of the circle?*

A. This question is apt because it addresses one of two principal methods of measuring the world. The first is a measurement of a finite, non-repeating aspect of the world. We can use scalar or exponential measure, like feet measuring length, or decibels measuring volume of sound. The second is the measure of cyclical aspects, like the circle and time. For the first aspect, divisibility is not quite as important as for the second. A cyclical aspect has a definite implied "whole", the cycle itself, that demands incorporation in any consideration of how to measure it.

There are three principal ways a duodecimal culture might divide the circle (or anything cyclical. For simplicity, let's focus on the circle). The circle can be divided into 2π radians in any number base. A duodecimal division of the circle, whether "pure" (i.e., strictly in parts that are negative powers of the dozen) or multiples of the dozen (e.g., in two dozen parts), is convenient and would seem to be sufficient for most purposes. Finally, the circle can be divided using an auxiliary number base, just as the number 360 is used in decimal.

The first option requires no further explanation. Clearly, the constant π would have a different appearance. Dozenally, π = 3.184 809 493 Ɛ91 866 457 3Ӽ6 212, (rounded to two dozen negative places) still an irrational number.

The other two benefit from some explanation. The DSA has an article called "Validating the Dozenal Measure of Angle" in its forthcoming *Duodecimal Bulletin* (Vol. 50; №. 1), retrievable at http://www.dozenal.org/articles/db50111.pdf. Some of that article is gleaned to help explain why a dozenally-divided circle is sufficient for most purposes in today's civilization.

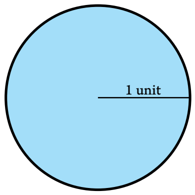


Figure 8.1. The unit circle, a circle with a radius of one unit of measure. This is the basis for this study of important angles.

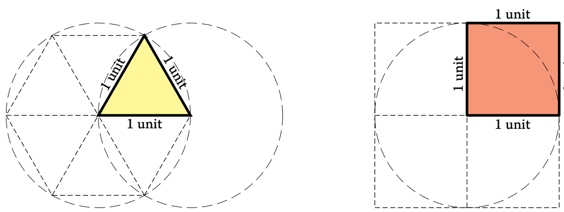


Figure 8.2. The equilateral triangle (left) and the square (right).

Figure 8.3. Bisecting the equilateral triangle and the square.

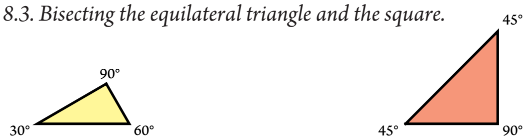


Figure 8.4. One half of the equilateral triangle and the square.

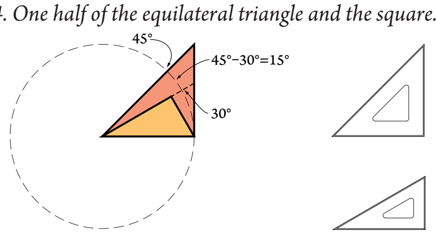
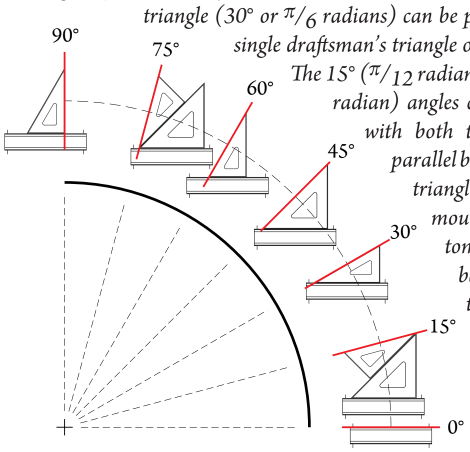


Figure 8.5. Difference between the shapes in Figure 8.4 are the basis of drafting triangles.

Figure 8.6. Two drafting triangles (the bisected cardinal shapes) and a parallel bar or T-square can be used to construct all basic angles (the two dozen angles at 15° or $\pi/12$ radian increments). The straight angle at 0° or 180° (0 or π radians) is constructed using the parallel bar or T-square alone. A right angle can be produced using the parallel bar or T-square and the perpendicular side of either of the drafting triangles. The angles of an equilateral triangle (60° or $\pi/3$ radians) or the bisector angles of either the square (45° or $\pi/4$ radians) or the equilateral triangle (30° or $\pi/6$ radians) can be produced using a single draftsman's triangle on a parallel bar. The 15° ($\pi/12$ radian) or 75° ($5\pi/12$ radian) angles can be produced with both triangles on the parallel bar. If the drafting triangles are flipped or mounted on the bottom of the parallel bar or T-square, the angles in the other quadrants of the circle can be drafted.



Strictly Dozenal Division of the Circle

We begin by examining the simple fact that a circle can be produced by fully rotating a line segment in a plane about one of its points (see Figure 8.1). Such a unit circle is the basis for all measurements of angle.

The first regular (two-dimensional) polygon many people will think of is the square, which can be made by propagating a line of a given length ℓ in a direction perpendicular to its length for the same length ℓ (See Figure 8.2). The resultant figure has four equal sides with four equal, right angles. This figure can be rotated so that exactly 4 such figures contain the circle in Figure 8.1—such a circle possesses a radius of ℓ. The square can be copied and tiled to fill up an infinite (Euclidean) two-dimensional plane. Graph paper demonstrates that the tiling of squares can be very handy. The Cartesian coordinate system with its familiar x and y axes, the street grids of cities like Chicago and Phoenix, the arrangement of columns in a big box store, all employ orthogonal arrangements of elements. Most of the built environment is based on the right angle. Our homes, offices, factories, streets, and cities commonly employ orthogonal geometry. Thus the right angle, a division of the circle into quarters, an angle of 90° ($\pi/2$ radians), is perhaps the most important division of the circle in everyday life.

The simplest regular polygon is a triangle, shown at left in Figure 8.2. We can construct a triangle having three sides of equal length and three corners with the same angles, thus an equilateral triangle, simply by placing a compass at one end of a line, drawing a circle as in Figure 8.1, then doing the same at the other end. We can draw straight lines from the intersections to both ends of the first lines to obtain a triangle. An equilateral triangle, if we were to copy it and cut it out, can be used to fill the circle in Figure 8.1: exactly six equilateral triangles with a common vertex can fill the circle. In fact, we can tile the equilateral triangle to fill up an infinite plane just like the square. We can make equilateral triangle "graph paper". In trigonometry, the cosine of 60° ($\pi/3$ radians) is exactly ½. Because the equilateral triangle is the simplest regular polygon, because it can fill two-dimensional planes, and because precisely 6 equilateral triangles can fill a circle, it follows that such a 3-sided figure is important. Its geometry is thus important. The angles we've generated are all 60° ($\pi/3$ radians), $1/6$ of a full circle. So dividing a circle into six equal angles is an important tool.

We can observe the importance of triangles in general and equilateral triangles in particular in our everyday society. Structural engineers design trusses, bar joists, and space frames with an equilateral arrangement, because the equilateral triangle is the stablest two-dimensional figure. Because its sides are equal, it can be mass-manufactured. The equilateral triangle is perhaps not as apparent as arrangements made with right angles (orthogonality), but it is important in the building of our everyday structures.

For the sake of this answer, we'll call the equilateral triangle and the square "cardinal shapes".

Figure 8.3 shows these cardinal shapes bisected (cut in half). There are three ways to bisect an equilateral triangle using one of its points, which are congruent if we rotate the triangle 120°

($2\pi/3$ radians). There are two ways to bisect a square using one of its points, which are congruent if we rotate the square 90° ($\pi/2$ radians). When we bisect an equilateral triangle, we obtain a right triangle with angles that measure 30°, 60°, and 90° ($\pi/6$, $\pi/3$, $\pi/2$ radians); see Figure 8.4. When we bisect a square, we obtain a right triangle with angles measuring 45° and 90° ($\pi/4$ and $\pi/2$ radians). These bisections are important because they relate a corner of an equilateral triangle with the midpoint of its opposite side, or the diametrically-opposed corners of a square to one another. In trigonometry, the sine of 30° ($\pi/6$ radians) is exactly ½. Thus, 30° and 45°, one dozenth and one eighth of a circle, respectively, are of secondary importance. We'll call the bisected equilateral triangle and the diagonally bisected square the "bisected cardinal shapes" for the sake of this answer.

Figure 8.5 shows that the difference between the bisecting angles of the cardinal shapes is 15° ($\pi/12$ radians), one two-dozenth of a circle. Using 15° or one two-dozenth of a circle as a snap-point, one can construct any incidence of the bisecting angles of a square or equilateral triangle. In fact, before the advent of computer-aided design and drafting (CADD), draftsmen commonly used a pair of "45°" and "30°–60°" drafting triangles, along with a T-square or parallel bar precisely to obtain the common angles which are two dozenths of a circle (see Figure 8.6). Thus, it is not by dozenalist design but sheer utility that the two-dozenth of a circle, or 15° ($\pi/12$ radian) angle is deemed important.

To be sure, other regular polygons can be drawn. The pentagon appears in regular three dimensional polyhedra, in the dodecahedron and the icosahedron, and in their symmetries. Geometrically, it is an important figure, however it cannot tile two dimensional space, and is not in common use in everyday life. The fact that most Americans can name precisely one building that is shaped like a pentagon contributes to the case that pentagonal arrangements are a curiosity because they are rare. The hexagon is important: we can see in Figure 8.2 that the outline of the group of triangles inscribed in the left circle is a regular hexagon. Thus the hexagon can tile two dimensional planes. Role playing games in the eighties employed the hexagonal grid system. There are some curious places (the Nassau Community College campus, the Price Tower in Bartlesville, Oklahoma, and other Frank Lloyd Wright buildings) which are arranged in a triangular-hexagonal geometry. The geometry of the hexagon is corollary to that of the equilateral triangle. There certainly are many regular polygons in use in human civilization and apparent in nature, but the commonest and most important geometries appear to be linked to the equilateral triangle and the square.

Other angles are important. On a map, we commonly hear "north northwest" or "east southeast", the sixteen partitions of the circle. The 22½° ($\pi/8$ radian) angle, one sixteenth of a circle, is important, perhaps more significant than 15° divisions in cartography. The sixteenth of a circle shouldn't be ignored for this reason.

The decimal division of the circle into thirty dozen degrees is a quite handy tool (See Figure 8.7). Each of the two dozen angles related to the equilateral triangle and the square (let's

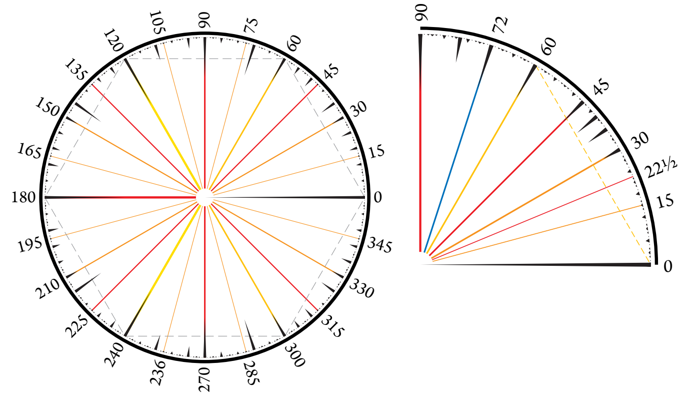


Figure 8.7. The degree system used in decimal civilization (left). The system was set up by our forefathers under a sexagesimal number base. The system survives to this day and continues to be used perhaps because all the basic angles are represented by decimal semiround or round numbers. At right, a closer examination of some key angles under the degree system. The right angle and the square bisector angle are 90° and 45° respectively. The equilateral angle and its bisector are 60° and 30° respectively. The difference betweeen the square and equilateral bisectors is 15°. The sixteenth of a circle is 22½° or 22°30′. The fifth of a circle measures 72°

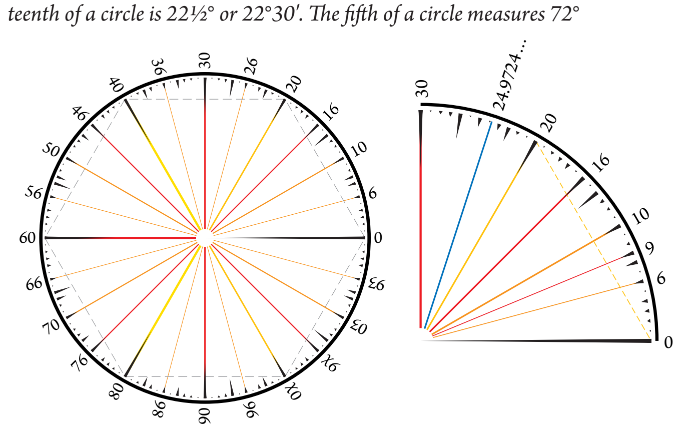


Figure 8.8. The circle divided into "temins", a gross temins to a full circle (left). The unitary relationship to the full circle is maintained, the notation for each of the basic angles is simpler, and the number of radians can be quickly determined by dividing the temins by six dozen, then multiplying by π. At right, a closer examination of some key angles expressed as temins. The right angle and the square bisector angle are 30^t and 16^t respectively. The equilateral angle and its bisector are 20^t and 10^t respectively. The difference betweeen the square and equilateral bisectors is 6^t. The dozen-fourth of a circle is simply 9^t. The fifth of a circle is 24;9724… ^{t}, a repeating digital fraction.

call these two dozen angles "basic angles") are resolved in the system of degrees without fractions. The system neatly accommodates fifths and tenths of a circle, although these are comparatively rarely used.

Under a dozenal system, we may discover that using a strictly dozenal division of the circle, perhaps using "temins" (perhaps abbreviated ^{t}) for convenience, neatly accommodates all the basic angles as well as the sixteenths of a circle without fractions (See Figure 8.8). The basic angles are simply multiples of 6^t. The sixteenths of a circle are multiples of 9^t. If we desire to "unify" the basic and the sixteenth-circles into one system, we might regard 3^t (7½°, $\pi/24$ radian) as the dozenal

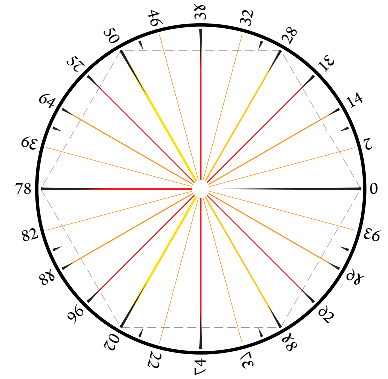


Figure 8.9. Hexadecimal auxiliary base, "f0" = decimal 240

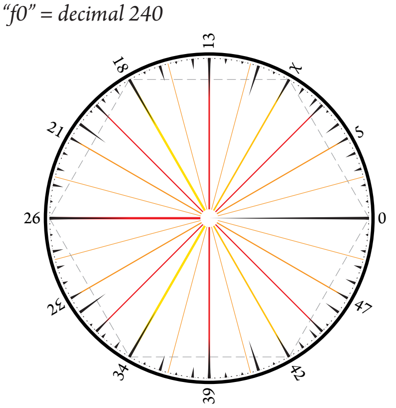


Figure 8.10. Dozenal auxiliary base, "50" = decimal 60

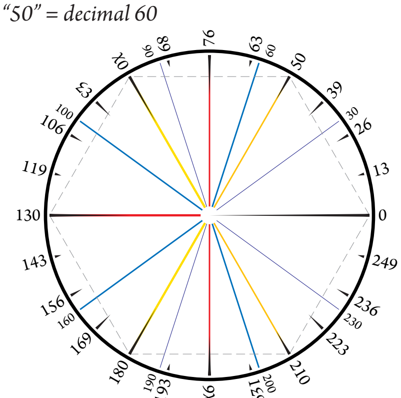


Figure 8.11. Dozenal auxiliary base, "260" = decimal 360

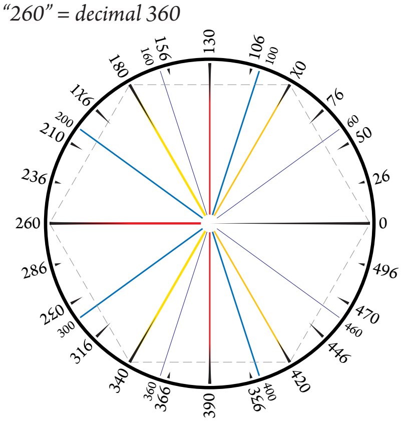


Figure 8.12. Dozenal auxiliary base, "500" = decimal 720

"basic angle". We surrender the fifths (pentagonal symmetry) and tenths to repeating fractions, but maintain a fairly simple measure of the commonest angles. A strictly dozenal division of a circle, as presented by our Founders, thus appears sound and sufficient for everyday use.

Auxiliary Bases

In today's world, the common division of the circle into 360 degrees (°) facilitates the division of the circle into "clean" thirds, sixths, and twelfths. [60]

An "auxiliary base" is one that we use to improve divisibility for cyclical measurements in a given number base. Let's call the number base that we assume everyone is generally using the "general base" or "the number base in play". In today's world, that would be the decimal.

The auxiliary bases are an ideal tool for bases that are "deficient" in a given required factor, like decimal or tetradecimal (base 14), and are less useful for highly factorable bases like dozenal. Due to the fact that sexagesimal is based on a highly factorable number (60) that is the product of the first three primes {2^2, 3, 5}, an auxiliary base is perhaps not required at all.

There are a few features of an auxiliary base we should examine. Firstly, the auxiliary base should be compact, perhaps involving no more than three figures as expressed by the number base in play, to be easily-borne in mind. There can be various scales of auxiliary bases giving a range of fineness. Consider decimal 12/24 for hours, 60 for minutes, and 360 for degrees of arc, each yielding levels of fineness of division.

Secondly, we are attempting to "resolve" both factors which are not present in a number base, such as three in decimal or five in dozenal, as well as maintain the "clean" native divisions furnished by the divisors of the general base, such as halves in both bases, dozenal thirds and decimal fifths. We may elect to "sacrifice" the "cleanliness" to "buy" resolution. Decimal sexagesimal does render clean fifths ($1/5$ = 0.2 = 12, a single-place fraction in the general base becomes a 2-place expression in the auxiliary base) but they are slightly less resolved than thirds ($1/3$ = 0.333... = 20, one significant digit in a 2-place auxiliary expression); the general base's resolution of quarters are maintained by the auxiliary ($1/4$ = 0.25 = 15).

Thirdly, the prime factorization of a number base "extracts" power from a highly factorable auxiliary base candidate. Generally, the superior highly composite numbers {2, 6, 12, 60, 120, 360, 2520, 5040,...} serve as decent auxiliary bases. Decimal, due to its deficiency regarding three, can sacrifice a little resolution of the less-important fifth to buy clean thirds, so the decimal auxiliary bases prove to be more compact. A base like hexadecimal is so rich in powers of two that the auxiliary base candidates need to include more multiplicity for the prime factor two than in other bases simply to overcome the hexadecimal "extraction". Hexadecimal users will likely be reluctant to sacrifice the clean eighths and sixteenths that enable "rolling" digits (doubling or halving reaching shifts the figure up or down a place every four iterations, e.g.: 3 → 6 → c → 18 → 30) but may be able to live with a less clean sixteenth to buy thirds. In octal and hexadecimal, one may be interested not only in resolution of thirds, but also fifths. Ultimately, the auxiliary bases that enjoy application are a societal choice; in decimal somehow we find sixty more convenient than one hundred twenty. It may not be apparent to us that a hexadecimal society would elect to give up clean eighths and sixteenths for resolution of thirds and fifths, implying an auxiliary base of hexadecimal "f0" = decimal 240 (see Figure 8.9), "3c0" = decimal 960; the large "f00" = 3840 may be better but is certainly finer.

The easiest thing to do to get a highly divisible auxiliary base in any general base is to multiply the base r by sixty. This delivers "sexagesimal cleanliness" to the base, e.g. clean halves, thirds, quarters, fifths, and sixths, etc. One can then elect to sacrifice some clean resolution of one factor, normally a native factor (otherwise why use an auxiliary if you torpedo one of its prime factors?), to optimize the size or achieve a target scale. Decimal uses unmodified sixty, giving clean halves, thirds, quarters, fifths, sixths, etc., "living with" a blunted quarter and fifth (with respect to sexagesimal). The long hundred (120) would yield a clean quarter but still requiring a less clean fifth, at the expense of grander scale. To get a totally clean fifth, we would need to use 600, which gives us another power of five, requiring no sacrifices, if sexagesimal cleanliness is the required standard.

Dozenal is taxed because it has a greater multiplicity of two and a compact, gapless prime factorization. Unmodified sexagesimal will prove deficient, yielding resolution of fifths at the expense of all the resolution of quarters, thirds, and even halves that users of dozenal will likely see as second nature. The use of the auxiliary bases we employ in decimal, such as 60 and 360, are unsatisfactory in duodecimal (see Figures 8.10 for 60 and 8.11 for 360). Half a circle would subtend 26; dozenal degrees (30 decimal), with five dozen degrees to the circle. This implies that a dozenal hour (if a dozenal society would use hours) would not likely get divided into sixty minutes. Half a circle of 130; degrees (180 decimal) would be fine, but 76; degrees (90 decimal) for a quarter circle would be hard to live with.

To guarantee sexagesimal cleanliness in base twelve, we can use 60*r*, thus duodecimal 500; (decimal 720, see Figure 8.12) to achieve a small scale auxiliary base. The problem we'll have with dozenal is the only factors we can sacrifice are the more useful third, half, and quarter, to buy fifths. In base ten, we can bargain our fifth to get a good third. So dozenal is kind of stuck with a relatively large scale auxiliary base, while those for decimal are relatively compact.

Summary

The use of duodecimal as the civilizational number base of computation would not affect the use of radians. A circle divided purely dozenally, that is, in a power of twelve, is sufficient to replace the 360° circle. The "clean" resolution of fifths and tenths (72° and 36°, respectively) would become more complicated (24.9724…° and 12.4972…°) but this might be tolerable. A better division of a circle would be a duodecimal 500;° circle (500; dozenal = 720 decimal). This would resolve all the angles that the decimal 360° circle, and still maintain duodecimally-round numbers for the other major angles.

9. *How if any would having a different system affect higher mathematics, above the fundamentals?*

Using a different number base would have a limited effect on higher mathematics. Arithmetic is different, but operates using the same rules. Constants would need conversion. Calculus would not be any different.

10. *If the world of numbers was to start over, which number system would be best for humans to choose to use? (doesn't have to be decimal, or duodecimals, could be any)*

[The following represents the author's opinion and is not necessarily the opinion of the Dozenal Society of America].

I think the world actually had it right as it started. I wouldn't have changed the early development of numeracy in the world, only the way it was represented. The world did not begin by using sexagesimal (base sixty), but that base was one of the earliest technologies used. Our forefathers had it right; so right that we still use sixty minutes in an hour, sixty seconds in a minute, and a circle of 360 degrees!

One of the earliest number systems the world produced was the Mesopotamian sexagesimal system. It had sixty digits that were composed of 6 "decade" figures and ten "unit figures" shown in Figure 10.1 below:

	0□	1□	2□	3□	4□	5□
□0	0	10 𒌋	20 𒎙	30 𒌍	40 𒐏	50 𒐐
□1	1 𒁹	11 𒌋𒁹	21 𒎙𒁹	31 𒌍𒁹	41 𒐏𒁹	51 𒐐𒁹
□2	2 𒈫	12 𒌋𒈫	22 𒎙𒈫	32 𒌍𒈫	42 𒐏𒈫	52 𒐐𒈫
□3	3 𒐈	13 𒌋𒐈	23 𒎙𒐈	33 𒌍𒐈	43 𒐏𒐈	53 𒐐𒐈
□4	4 𒐉	14 𒌋𒐉	24 𒎙𒐉	34 𒌍𒐉	44 𒐏𒐉	54 𒐐𒐉
□5	5 𒐊	15 𒌋𒐊	25 𒎙𒐊	35 𒌍𒐊	45 𒐏𒐊	55 𒐐𒐊
□6	6 𒐋	16 𒌋𒐋	26 𒎙𒐋	36 𒌍𒐋	46 𒐏𒐋	56 𒐐𒐋
□7	7 𒐌	17 𒌋𒐌	27 𒎙𒐌	37 𒌍𒐌	47 𒐏𒐌	57 𒐐𒐌
□8	8 𒐍	18 𒌋𒐍	28 𒎙𒐍	38 𒌍𒐍	48 𒐏𒐍	58 𒐐𒐍
□9	9 𒐎	19 𒌋𒐎	29 𒎙𒐎	39 𒌍𒐎	49 𒐏𒐎	59 𒐐𒐎

Figure 10.1 The ancient Mesopotamian sexagesimal digits were composed of five decade-figures and nine unit-figures. With these fourteen figures, sixty unique digits were composed. One may find the corresponding example of an ancient Mesopotamian digit by finding the decade-figure in the header row, then using the leftmost column, finding the unit-figure. By running a finger to the column of the high rank figure, the unique sexagesimal digit can be identified. Example: decade-figure 5□ and unit-figure □4 renders the digit-54: 𒐐𒐉.

My opinion is that this system is even stronger than duodecimal, but our forefathers' numerals weren't. (See Figure 10.1 [61]) I think that the senary (base six) decade-figures and the decimal unit-figures of what I call "6-on-ten" sexagesimal may rival duodecimal numeration. We are familiar with the 6-on-ten arrangement, as it is seen on our digital clocks, in degrees-minutes-seconds notation for angles. If we were to simply write numbers in 6-on-ten without semicolons, we would have a fine tool for general intuitive human computation. Example, take the time for sunset today: 4:40 PM. If we wrote "1640" we would then be representing the time in 6-on-ten sexagesimal. Counting from the least significant digit, every odd digit is a decimal unit-figure, while every even digit is a senary decade-figure. As long as we keep that in mind, we can perform arithmetic using the power of a number base that is even stronger than duodecimal. The problem is that people may not be in the habit of regarding one digit as "senary" and the other as "decimal" and could produce error. Most people would be confused by the arrangement. We could stack the decade-figures on the unit-figures like this: $^{1}_{6}{}^{4}_{0}$. This is actually closer to having pure sexagesimal digits, and is a modern analog to what our forefathers were doing five-thousand-plus years ago in Mesopotamia: they would have written "$\overset{1}{6}\overset{4}{0}$" as "𒌋𒐊𒐏". Also note that the sexagesimal number $\overset{1}{6}\overset{4}{0}$ = 16(60) + 40 = decimal 1000.

I have used "pure" sexagesimal extensively using the following "arqam" numerals [62, 63] invented in 1992:

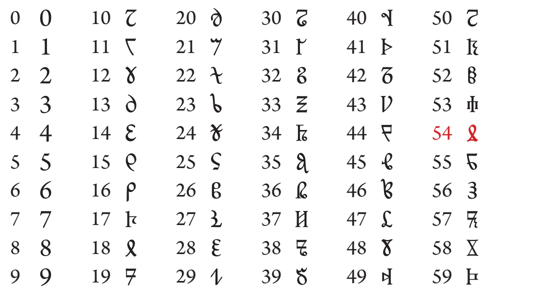


Figure 10.2. The first sixty "Arqam" numerals, used to represent "pure" sexagesimal.

Using the example from above, I can write the decimal number 1000 as ϸ𐎗. So we can write the sexagesimal version of decimal 1000 in the following ways:

1000	16:40	1640	$\overset{1}{6}\overset{4}{0}$	ϸ𐎗	𒌋𒐊𒐏

I use these sixty numerals because they express sexagesimal in a "pure" manner, without decimal "contamination" of the properties of sexagesimal from the 6-on-ten arrangement. If I write the multiples of six in stacked 6-on-ten sexagesimal, I get the familiar-looking decimal sequence. But using arqam, I have single digits that divorce the decimal way of seeing sixes. Remember that 3 is a decimal semitotative, but in base sixty, six is a divisor.

0	6	$\overset{1}{2}$	$\overset{1}{8}$	$\overset{2}{4}$	$\overset{3}{0}$	$\overset{3}{6}$	$\overset{4}{2}$	$\overset{4}{8}$	$\overset{5}{4}$
0	6	ℵ	ℓ	ϒ	Շ	ℛ	Ɵ	Ծ	ℓ

Duodecimal has an advantage over decimal because it has more divisors. The particular small, consecutive prime dozenal divisors {2, 3} yield greater numbers of dozenal regular numbers while minimizing resistance from totatives. Sexagesimal has a dozen divisors {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60} compared to only six for twelve {1, 2, 3, 4, 6, 12}. We used 256 as a benchmark to compare the regular numbers of bases ten, twelve, and sixteen. Here is a list of the 52 sexagesimal regular numbers less than 256 [64]:

{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50, 54, 60, 64, 72, 75, 80, 81, 90, 96, 100, 108, 120, 125, 128, 135, 144, 150, 160, 162, 180, 192, 200, 216, 225, 240, 243, 250, 256}

Recall that decimal sports 21, dozenal 28, hexadecimal just 9 regular digits below 256.

Nothing lasts forever and empires fall. I visited the Metropolitan Museum of Art in New York City last year. There were records that had the date "3100 BC" on it, so I thought "3100 years ago." *No, 5100 years ago!* The tablet Plimpton 322 [65], produced around 3800 years ago, on which is scribed some sexagesimal numbers in cuneiform. The tablet implies that the Babylonians were familiar with the Pythagorean theorem well before Pythagoras! There are very many records dating from this time. Base sixty was the ancient Mesopotamian "decimal" in that it dominated society, and the users were sometimes pondering relatively complex mathematics. I believe we ought never to have changed what our forefathers started.

11. *What numerals would be used by a society that used duodecimal rather than decimal?*

Clearly a number base $r > 10$ will need more than ten numerals if we are using standard positional notation, where one numeral (symbol) represents one digit $0 \le n < r$. The digit "0" represents congruency with the number base r, and rarely represents actual zero (i.e., when "0" stands alone as an integer) [66]. In base twelve, we need twelve numerals. We may use the decimal numerals {0, 1, 2, …, 9} as they are, and simply append two "transdecimal" numerals that symbolize digit-ten and digit-eleven: this is referred to by the Dozenal Society of America (DSA) as the "Principle of Least Change"[67]. Here are six examples [68]:

Least Change	0	1	2	3	4	5	6	7	8	9	10	11
Alphanumeric	0	1	2	3	4	5	6	7	8	9	a	b
Sir Issac Pitman	0	1	2	3	4	5	6	7	8	9	↊	↋
W. A. Dwiggins	0	1	2	3	4	5	6	7	8	9	χ	Ɛ
"Bell" via Churchman	0	1	2	3	4	5	6	7	8	9	Ӿ	♯
R. Greaves / D. James	0	1	2	3	4	5	6	7	8	9	ɫ	↋
M. De Vlieger "Arqam"	0	1	2	3	4	5	6	7	8	9	↊	↋

Alternatively, we can create an entirely new set of numerals for dozenal such as six examples of the "Principle of Separate Identity" below [68]:

Separate Identity	0	1	2	3	4	5	6	7	8	9	10	11
F. Ruston	0	/	Λ	Π	⋜	X	P	∠	Λ	Π	⋛	X
R. J. Hinton	Ø	P	ß	ℰ	ɕ	ȣ	ȣ	ℵ	ℵ	♈	ɟ	ℵ
P. D. Thomas	Ø	⅃	ϟ	Ψ	ƚ	λ	Ӿ	¥	ǂ	Ʊ	∠	⊦
S. Ferguson (1)	0	1	ħ	ℎ	ƀ	₽	Ꝝ	Λ	ḿ	V	F	Ɛ
W. Lauritzen (1)	o	ɵ	ɵ	ɵ	ɵ	ɵ	ȯ	σ	ɑ	ɋ	ɒ	ʊ
M. De Vlieger (2)	0	1	2	3	3	∂	8	6	Ɛ	Ɛ	↊	↋

A number base r may produce exactly r numerals using elements. The following example has a numeral for zero (▯), then builds the numerals 1 through 5 using horizontal strokes (_), then adds a vertical stroke (|) to each digit $0 < n \le 5$ to produce each digit $n \ge 6$. There are thus only three elements that produce the dozen digits necessary to represent base twelve.

Elements	0	1	2	3	4	5	6	7	8	9	10	11
R. Marino	▯	_	=	≡	≣	≣	ǀ	∟	∟	⊨	⊨	⊨

This scheme is similar to the vigesimal (base-twenty) numerals used by the Mayans or the sexagesimal (base-sixty) numerals used by the ancient Mesopotamians.

The DSA does NOT endorse any particular symbols for the digits ten and eleven. For uniformity in publications we use Dwiggins "dek" (χ) for ten and his "el" (Ɛ) for eleven. Whatever symbols are used, the numbers commonly called "ten", "eleven" and "twelve" are pronounced "dek", "el" and "dough" in Dwiggin's duodecimal system.

When it is not clear from the context whether a numeral is decimal or dozenal, we use a period as a unit point for base ten and a semicolon, or Humphrey point, as a radix point for base twelve.

Thus ½ = 0;6 = 0.5, 2⅔ = 2;8 = 2.666..., 6⅜ = 6;46 = 6.375.

The Dozenal Society of Great Britain (DSGB) uses Sir Issac Pitman's transdecimal numerals, (Ꮗ) for digit-ten and (Ɛ) for digit-eleven.

The DSA welcomes experimentation with numeral forms, providing various resources at its website [68, 69, 70] that intend to aid those considering producing their own numerals. Take a look at this webpage at the DSA website for more information on duodecimal numerals:

http://www.dozenal.org/articles/numerals.html.

Sources:

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I Neugebauer, Otto., *The Exact Sciences in Antiquity*, Mineola, NY: Dover, 1969. [2nd ed., Providence, RI: Brown University Press, 1957.]

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K Flegg, Graham, *Numbers through the Ages*. London: Macmillan Publishers Ltd., 1989.

L Hardy, G. H., Seshu Aiyar, P. V., Wilson B. M., *Collected Papers of Srinivasa Ramanujan*. Providence, RI: AMS Chelsea Publishing, 1927.

Divisors:

1 Glaser 1971, Chapter 4, "The Rest of the 1700s", Section "Duodecimal versus a Decimal Metric System", page 69, specifically: "Any advantages of base 12 would be due to its being richer in divisors than 10 and certain common fractions such as ⅓ and ¼ would have simpler equivalents in base 12 notation."

2 Dudley 1969, Chapter 14, "Duodecimals", page 114, specifically: "Any base would serve as well as 12 to give practice, but some parts of arithmetic—notably decimals—are nicer in base 12 than they are in base 10. Besides, there is a good deal of twelveness in everyday life: items are measured by the dozen and gross, there are 12 months in a year, 12 inches in a foot, half a dozen feet in a fathom, two dozen hours in a day, and 30 dozen degrees in a circle. The reason for this abundance of twelves is the easy divisibility of 12 by 3, 4, and 6; we want to make such divisions much more often than we want to divide things by 5. … counting by the dozens is manifestly better."

3 Menninger 1957, "Number Sequence and Number Language", "Our Number Words", "The Number Twelve as the Basic Unit of the Great Hundred", pages 156-7, specifically: "But why precisely 12? The importance of this number in the lives of common people, in commercial transactions, and in legal affairs is probably due to its easy divisibility in so many ways… The commonly used fractions of …the shilling could all be expressed in terms of whole numbers of pennies" and the ensuing example.

4 Glaser 1971, Chapter 2, "Before Leibniz", Section "Blaise Pascal", page 17, specifically: "…the duodecimal system (a most convenient one indeed)…"

5 Glaser 1971, Chapter 9, "Summary", Section "Which base is best?", page 170, specifically: "Ten was a poor choice, argued Pascal in 1665. He and many later writers looked for *richness in divisors* (and consequent simple fractions) and indicated their preference for 12 as the base of our numeration system."

6 Glaser 1971, Chapter 4, "The Rest of the 1700s", Section "Lagrange, Laplace, Lamarque, and Legendre", page 70, specifically: "…Laplace, however, who lectured at [the École Normale in Paris in 1795], let it be known, that among all possible bases, one would be preferable that is not too large and has a great many divisors, and that it was base 12 that met these requirements."

7 Menninger 1957, "Number Sequence and Number Language", "Our Number Words", "The Roman Duodecimal Fractions", page 158, specifically: "The Roman fractions were originally based on a system of equivalent weights; 1 *as* (or pound) = 12 *unciæ* (ounces). … The underlying tendency of all subdivisions of measures is to avoid fractions and to express them instead as whole numbers of smaller units."

8 Menninger 1957, "Number Sequence and Number Language", "Our Number Words", "The Roman Duodecimal Fractions", page 158, specifically: "In their computations the Romans used no fractions other than these duodecimal fractions."

9 Butterworth 1999, Chapter 2, "Everybody Counts", Section 4, "Basic bases", pages 69–70, specifically: "English contains residues of other bases that have been used in parallel with base-10, and which historically, may have preceded it in Europe. Base-12 was used for money (12 pennies to the shilling), weight of precious metals (12 Troy ounces to the pound), length (12 inches to the foot), and quantity (dozen, gross). The word *twelve* reflects its origins much less transparently than the 'teen' words, which are still clearly in the form 'unit + ten'. The division of a day and night, and our clock face, into 12 hours has an ancient, Egyptian origin, and is possibly the source of our twelve-counting."

10 Ore 1948, Chapter 5, "The aliquot parts", page 86, specifically, Equation 5-2: "For any divisor d of N one has $N = dd_1$, where d_1 is the divisor paired with d."

11 Flegg 1989, Chapter 5, "Fractions and Calculation", Section "Natural and unit fractions", page 131, specifically: "In spoken English certain simple fractions like ½ or ¼ or ¾ have special names. We do not say 'one-fourth' but one quarter, and ½ is pronounced as one half. The French have a special name tiers for ⅓. Fractions of this kind, which serve the purposes of everyday life, may be called natural fractions. In ancient Egypt the situation was similar. The Egyptians had special words for the natural fractions ½, ⅓, ⅔, ¼ and ¾."

12 Neugebauer 1957, Chapter iv, "Egyptian Mathematics and Astronomy", Section "Natural and unit fractions", page 74, specifically: "…the separation of all unit fractions into two classes, 'natural' fractions and 'algorithmic' fractions … As 'natural' fractions I consider the small group of fractional parts which are singled out by special signs or special expressions from the very beginning, like [⅔, ⅓, ½, and ¼]. These parts are individual units which are considered basic concepts on an equal level with the integers. They occur in everywhere in daily life, in counting and measuring."

13 Weisstein, Eric W. "Divisor Function", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/DivisorFunction.html >, specifically for the $\sigma_0(r)$ and $\sigma_1(r)$ notation, equivalent to Dudley's $d(n)$ and $\sigma(n)$, respectively, and Ore's $\nu(n)$ and $\sigma(n)$, respectively.

14 Sloane, Neil J. A. "A000005 … the number of divisors of *n*", Online Encyclopedia of Integer Sequences. Retrieved November 2011, < http://oeis.org/A000005 >.

15 Hardy, G. H., Seshu Aiyar, P. V., Wilson B. M., *Collected Papers of Srinivasa Ramanujan*. Providence, ri: AMS Chelsea Publishing, 1927. "Highly Composite Numbers", Section III "The Structure of Highly Composite Numbers", Item 6, page 86, specifically "A number *N* may be said to be a highly composite number, if $d(N') < d(N)$ for all values of N' less than *N*."

16 Sloane, Neil J. A. "A002182 Highly composite numbers", Online Encyclopedia of Integer Sequences. Retrieved November 2011, < http://oeis.org/A002182 >.

17 Hardy, G. H., Seshu Aiyar, P. V., Wilson B. M., *Collected Papers of Srinivasa Ramanujan*. Providence, ri: AMS Chelsea Publishing, 1927. "Highly Composite Numbers", Section IV "Superior Highly Composite Numbers", Item 32, page 111, specifically "A number *N* may be said to be a superior highly composite number, if there is a positive number ε such that $d(N)/N^{\varepsilon} \geq d(N')/(N')^{\varepsilon}$ for all values of N' less than *N*, and $d(N)/N^{\varepsilon} > d(N')/(N')^{\varepsilon}$ for all values of N' greater than *N*."

18 Sloane, Neil J. A. "A002201 Superior Highly Composite Numbers", Online Encyclopedia of Integer Sequences. Retrieved November 2011, < http://oeis.org/A002201 >.

Other Duodecimal Benefits:

19 Weisstein, Eric W. "Digit", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/Digit.html >.

20 Weisstein, Eric W. "Numeral", Wolfram MathWorld. Retrieved August 2011, < http://mathworld.wolfram.com/Numeral.html >.

21 LeVeque 1962, Chapter 1, "Foundation", Section 1–5, "Radix representation", page 17, specifically: "Theorem 1–1. If *a* is positive and *b* is arbitrary, there is exactly one pair of integers *q*, *r* such that the conditions $b = qa + r$, $0 \leq r < a$, hold" and the ensuing proof.

22 LeVeque 1962, Chapter 1, "Foundation", Section 1–5, "Radix representation", page 18, specifically: "Theorem 1–2. Let *g* be greater than 1. Then each integer *a* greater than 0 can be represented uniquely in the form $a = c_0 + c_1 g + \ldots + c_n g^n$, where c_n is positive and $0 \leq c_n < g$ for $0 \leq m \leq n$" and the ensuing proof.

23 LeVeque 1962, Chapter 1, "Foundation", Section 1–5, "Radix representation", page 19, specifically: "…we can construct a system of names or symbols for the positive integers in the following way. We choose arbitrary symbols to stand for the digits (i.e., the nonnegative integers less than g) and replace the number $c_0 + c_1 g + \ldots + c_n g^n$ by the simpler symbol $c_n c_{n-1} \ldots c_1 c_0$. For example, by choosing *g* to be ten and giving the smaller integers their customary symbols, we have the ordinary decimal system … But there is no reason why we must use ten as the base or radix…"

24 Dudley 1969, Section 4, "Congruences", page 28, specifically: "Theorem 1. $a \equiv b \pmod{m}$ if and only if there is an integer *k* such that $a = b + km$" and the ensuing proof.

25 Ore 1948, Chapter 2, "Properties of numbers. Division", page 29, specifically: "Each number has the obvious decomposition $c = 1 \cdot c = (-1)(-c)$ and ± 1 together with $\pm c$ are called trivial divisors."

26 Hardy & Wright 2008, Chapter v, "Congruences and Residues", Section 5.1, "Highest common divisor and least common multiple", page 58, specifically: "If $(a, b) = 1$, *a* and *b* are said to be prime to one another or coprime. The numbers $a, b, c, \ldots, k$ are said to be coprime if every two of them are coprime. To say this is to say much more than to say that $(a, b, c, \ldots, k) = 1$, which means merely that there is no number but 1 which divides all of $a, b, c, \ldots, k$. We shall sometimes say '*a* and *b* have no common factor' when we mean that they have no common factor greater than 1, i.e. that they are coprime."

27 Jones & Jones 2005, Chapter 1, "Divisibility", Section 1.1, "Divisors", page 10, specifically: "Definition: Two integers *a* and *b* are coprime (or relatively prime) if $\gcd(a, b) = 1$ … More generally, a set of integers $a_1, a_2, \ldots$ are coprime if $\gcd(a_1, a_2, \ldots) = 1$, and they are mutually coprime if $\gcd(a_i, a_j) = 1$ whenever $i \neq j$. If they are mutually coprime then they are coprime (since $\gcd(a_1, a_2, \ldots) \mid \gcd(a_i, a_j)$), but the converse is false."

28 Jones & Jones 2005, Chapter 1, "Divisibility", Section 1.1, "Divisors", page 10, specifically: "Corollary 1.9: Two integers *a* and *b* are coprime if and only if there exist integers *x* and *y* such that $ax + by = 1$" and the ensuing proof.

29 LeVeque 1962, Chapter 2, "The Euclidean Algorithm and Its Consequences", Section 2–2, "The Euclidean algorithm and greatest common divisor", page 24, specifically: "(e) if a given integer is relatively prime to each of several others, it is relatively prime to their product." and the ensuing example.

30 Weisstein, Eric W. "Relatively Prime", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/RelativelyPrime.html >.

31 Weisstein, Eric W. "Totative", Wolfram MathWorld. Retrieved February 2011, < http://mathworld.wolfram.com/Totative.html >.

32 Ore 1948, Chapter 5, "The aliquot parts", pages 109–110, specifically: "5-5. Euler's function. When *m* is some integer, we shall consider the problem of finding how many of the numbers $1, 2, 3, \ldots, (m - 1), m$ are relatively prime to *m*. This number is usually denoted by $\varphi\,(m)$, and it is known as Euler's φ-function of *m* because Euler around 1760 for the first time proposed the question and gave its solution. Other names, for instance, indicator or totient have been used."

33 Jones & Jones 2005, Chapter 5, "Euler's Function", Section 5.2, "Euler's function", page 85, specifically: "Definition: We define $\phi(n) = |U_n|$, the number of units in $\mathbb{Z}_n$ … this is the number of integers $a = 1, 2, \ldots, n$ such that $\gcd(a, n) = 1$" and the ensuing proof and example.

34 Hardy & Wright 2008, Chapter I, "The Series of Primes", Section 1.2, "Prime numbers", page 2, specifically: "It is important to observe that 1 is not reckoned as a prime."

35 Jones & Jones 2005, Chapter 2, "Prime Numbers", Section 2.1, "Prime numbers and prime-power factorisations", page 20, specifically: "Note that 1 is not prime."

36 Dudley 1969, Section 2, "Unique factorization", page 10, specifically: "Note that we call 1 neither prime nor composite. Although it has no positive divisors other than 1 and itself, including it among the primes would make the statement of some theorems inconvenient, in particular, the unique factorization theorem. We will call 1 a unit."

37 De Vlieger 2011, *Neutral Digits*, publication pending.

38 Ore 1948, Chapter 13, "Theory of Decimal Expansions", page 316, specifically: "In general, let us say that a number is *regular* with respect to some base number *b* when it can be expanded in the corresponding number system with a finite number of negative powers of *b*. … one concludes that the regular numbers are the fractions $r = p/q$, where *q* contains no other prime factors other than those that divide *b*."

39 Weisstein, Eric W. "Regular Number", Wolfram MathWorld. Retrieved February 2011, <http://mathworld.wolfram.com/RegularNumber.html>.

40 Hardy & Wright 2008, Chapter ix, "The Representation of Numbers by Decimals", Section 9.6, "Tests for divisibility", page 144–145.

41 Hardy & Wright 2008, Chapter ix, "The Representation of Numbers by Decimals", Section 9.6, "Tests for divisibility", page 145.

42 Hardy & Wright 2008, Chapter ix, "The Representation of Numbers by Decimals", Section 9.6, "Tests for divisibility", pages 146–147, specifically: "A number is divisible by 2 if its last digit is even. More generally, it is divisible by 2^{ν} if and only if the number represented by its last ν digits is divisible by 2^{ν}. The reason, of course is that $2^{\nu} \mid 10^{\nu}$; and there are similar rules for 5 and 5^{ν}."

43 Hardy & Wright 2008, Chapter ix, "The Representation of Numbers by Decimals", Section 9.6, "Tests for divisibility", page 147, specifically: "Since $10 \equiv -1 \pmod{11}$, we have $10^{2r} = 1$, $10^{2r+1} = -1 \pmod{11}$, so that $A1.10^{s} + A2.10^{(s-1)} + \ldots + A_s.10 + A_{(s+1)} \equiv A_{(s+1)} - A_s + A_{(s-1)} \ldots \pmod{11}$. A number is divisible by 11 if and only if the difference between the sums of its digits of odd and even ranks is divisible by 11."

44 Hardy & Wright 2008, Chapter ix, "The Representation of Numbers by Decimals", Section 9.6, "Tests for divisibility", page 147, specifically: "Next $10^{\nu} \equiv 1 \pmod{9}$ for every ν, and therefore $A1.10^{s} + A2.10^{(s-1)} + \ldots + A_s.10 + A_{(s+1)} \equiv A_1 + A_2 + \ldots + A_{(s+1)} \pmod{9}$. A fortiori this is true mod

3. Hence we obtain the well-known rule 'a number is divisible by 9 (or by 3) if and only if the sum of its digits is divisible by 9 (or by 3)'."

Duodecimal Education:

45 Glaser 1971, Chapter 5, "The Nineteenth Century", Section III "The third quarter", page 84.

46 "Isaac Pitman", Wikipedia. Retrieved November 2011, < http://en.wikipedia.org/wiki/Isaac_Pitman >.

47 Andrews, F. Emerson, "An Excursion in Numbers", *Atlantic Monthly*. Volume 154, October 1934. *See also* The Dozenal Society of America. Retrieved November 2011, < http://www.dozenal.org/articles/Excursion.pdf >.

48 Andrews, F. Emerson, "My Love Affair with Dozens", *Michigan Quarterly Review*. Volume xi, №. 2, Spring 1972. *See also* The Dozenal Society of America. Retrieved November 2011, < http://www.dozenal.org/articles/DSA-MyLoveAffair.pdf >.

49 Essig, Jean, *Douze notre dix futur* (French). Paris: Dunod, 1955.

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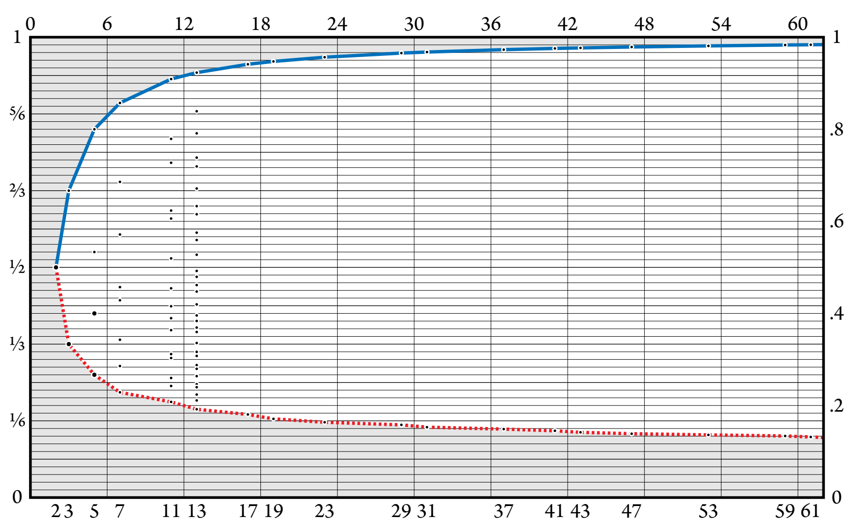


Figure 1.15. A plot to scale with $\phi(r)/r$ on the vertical axis versus the maximum distinct prime divisor p_{max} on the horizontal axis. The p_{max}-smooth numbers lie along a vertical line at each value of p_{max}. The boundary of minimum values of $\phi(r)/r$ defined by primorials is indicated by a broken red line. The boundary of maximum values of $\phi(r)/r$ defined by primes is shown in blue. All other composite numbers r that have p_{max} as the maximum distinct prime divisor inhabit the region between the boundaries. (See Figure 1.11 for detail at $2 \le p_{max} \le 13$.)

1	2	3	4	5	6	7	10
2	4	6	10	12	14	16	20
3	6	11	14	17	22	25	30
4	10	14	20	24	30	34	40
5	12	17	24	31	36	43	50
6	14	22	30	36	44	52	60
7	16	25	34	43	52	61	70
10	20	30	40	50	60	70	100

Base 8 (Octal)

1	2	3	4	5	6	7	8	9	χ	10
2	4	6	8	χ	11	13	15	17	19	20
3	6	9	11	14	17	1χ	22	25	28	30
4	8	11	15	19	22	26	2χ	33	37	40
5	χ	14	19	23	28	32	37	41	46	50
6	11	17	22	28	33	39	44	4χ	55	60
7	13	1χ	26	32	39	45	51	58	64	70
8	15	22	2χ	37	44	51	59	66	73	80
9	17	25	33	41	4χ	58	66	74	82	90
χ	19	28	37	46	55	64	73	82	91	χ0
10	20	30	40	50	60	70	80	90	χ0	100

Base 11

1	2	3	4	5	6	7	8	9	Շ	Ꞁ	ɤ	∂	10
2	4	6	8	Շ	ɤ	10	12	14	16	18	1Շ	1ɤ	20
3	6	9	ɤ	11	14	17	1Շ	1∂	22	25	28	2Ꞁ	30
4	8	ɤ	12	16	1Շ	20	24	28	2ɤ	32	36	3Շ	40
5	Շ	11	16	1Ꞁ	22	27	2ɤ	33	38	3∂	44	49	50
6	ɤ	14	1Շ	22	28	30	36	3ɤ	44	4Շ	52	58	60
7	10	17	20	27	30	37	40	47	50	57	60	67	70
8	12	1Շ	24	2ɤ	36	40	48	52	5Շ	64	6ɤ	76	80
9	14	1∂	28	33	3ɤ	47	52	5Ꞁ	66	71	7Շ	85	90
Շ	16	22	2ɤ	38	44	50	5Շ	66	72	7ɤ	88	94	Շ0
Ꞁ	18	25	32	3∂	4Շ	57	64	71	7ɤ	89	96	Շ3	Ꞁ0
ɤ	1Շ	28	36	44	52	60	6ɤ	7Շ	88	96	Շ4	Ꞁ2	ɤ0
∂	1ɤ	2Ꞁ	3Շ	49	58	67	76	85	94	Շ3	Ꞁ2	ɤ1	∂0
10	20	30	40	50	60	70	80	90	Շ0	Ꞁ0	ɤ0	∂0	100

Base 14

1	2	3	4	5	6	7	8	10
2	4	6	8	11	13	15	17	20
3	6	10	13	16	20	23	26	30
4	8	13	17	22	26	31	35	40
5	11	16	22	27	33	38	44	50
6	13	20	26	33	40	46	53	60
7	15	23	31	38	46	54	62	70
8	17	26	35	44	53	62	71	80
10	20	30	40	50	60	70	80	100

Base 9

1	2	3	4	5	6	7	8	9	χ	ε	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2ε	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2ε	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
ε	1χ	29	38	47	56	65	74	83	92	χ1	ε0
10	20	30	40	50	60	70	80	90	χ0	ε0	100

Base 12 (Duodecimal)

1	2	3	4	5	6	7	8	9	Շ	Ꞁ	ɤ	∂	ε	10
2	4	6	8	Շ	ɤ	ε	11	13	15	17	19	1Ꞁ	1∂	20
3	6	9	ɤ	10	13	16	19	1ɤ	20	23	26	29	2ɤ	30
4	8	ɤ	11	15	19	1ε	22	26	2Շ	2ε	33	37	3Ꞁ	40
5	Շ	10	15	1Շ	20	25	2Շ	30	35	3Շ	40	45	4Շ	50
6	ɤ	13	19	20	26	2ɤ	33	39	40	46	4ɤ	53	59	60
7	ε	16	1ε	25	2ɤ	34	3Ꞁ	43	4Շ	52	59	61	68	70
8	11	19	22	2Շ	33	3Ꞁ	44	4ɤ	55	5∂	66	6ε	77	80
9	13	1ɤ	26	30	39	43	4ɤ	56	60	69	73	7ɤ	86	90
Շ	15	20	2Շ	35	40	4Շ	55	60	6Շ	75	80	8Շ	95	Շ0
Ꞁ	17	23	2ε	3Շ	46	52	5∂	69	75	81	8ɤ	98	Շ4	Ꞁ0
ɤ	19	26	33	40	4ɤ	59	66	73	80	8ɤ	99	Շ6	Ꞁ3	ɤ0
∂	1Ꞁ	29	37	45	53	61	6ε	7ɤ	8Շ	98	Շ6	Ꞁ4	ɤ2	∂0
ε	1∂	2ɤ	4Ꞁ	4Շ	59	68	77	86	95	Շ4	Ꞁ3	ɤ2	∂1	ε0
10	20	30	40	50	60	70	80	90	Շ0	Ꞁ0	ɤ0	∂0	ε0	100

Base 15

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Base 10 (Decimal)

1	2	3	4	5	6	7	8	9	Շ	Ꞁ	ɤ	10
2	4	6	8	Շ	ɤ	11	13	15	17	19	1Ꞁ	20
3	6	9	ɤ	12	15	18	1Ꞁ	21	24	27	2Շ	30
4	8	ɤ	13	17	1Ꞁ	22	26	2Շ	31	35	39	40
5	Շ	12	17	1ɤ	24	29	31	36	3Ꞁ	43	48	50
6	ɤ	15	1Ꞁ	24	2Շ	33	39	42	48	51	57	60
7	11	18	22	29	33	3Շ	44	4Ꞁ	55	5ɤ	66	70
8	13	1Ꞁ	26	31	39	44	4ɤ	57	62	6Շ	75	80
9	15	21	2Շ	36	42	4Ꞁ	57	63	6ɤ	78	84	90
Շ	17	24	31	3Ꞁ	48	55	62	6ɤ	79	86	93	Շ0
Ꞁ	19	27	35	43	51	5ɤ	6Շ	78	86	94	Շ2	Ꞁ0
ɤ	1Ꞁ	2Շ	39	48	57	66	75	84	93	Շ2	Ꞁ1	ɤ0
10	20	30	40	50	60	70	80	90	Շ0	Ꞁ0	ɤ0	100

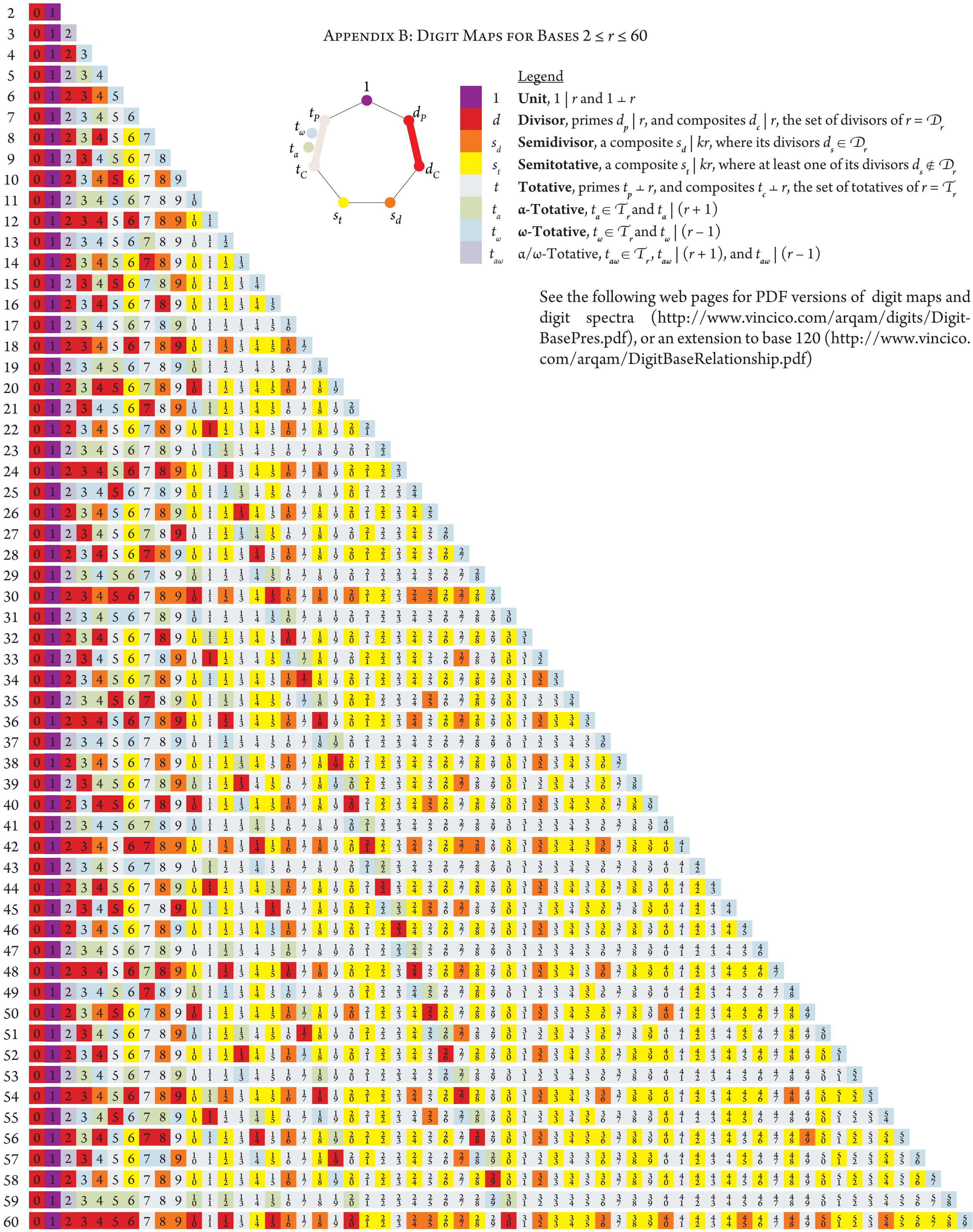
Base 13

1	2	3	4	5	6	7	8	9	a	b	c	d	e	f	10
2	4	6	8	a	c	e	10	12	14	16	18	1a	1c	1e	20
3	6	9	c	f	12	15	18	1b	1e	21	24	27	2a	2d	30
4	8	c	10	14	18	1c	20	24	28	2c	30	34	38	3c	40
5	a	f	14	19	1e	23	28	2d	32	37	3c	41	46	4b	50
6	c	12	18	1e	24	2a	30	36	3c	42	48	4e	54	5a	60
7	e	15	1c	23	2a	31	38	3f	46	4d	54	5b	62	69	70
8	10	18	20	28	30	38	40	48	50	58	60	68	70	78	80
9	12	1b	24	2d	36	3f	48	51	5a	63	6c	75	7e	87	90
a	14	1e	28	32	3c	46	50	5a	64	6e	78	82	8c	96	a0
b	16	21	2c	37	42	4d	58	63	6e	79	84	8f	9a	a5	b0
c	18	24	30	3c	48	54	60	6c	78	84	90	9c	a8	b4	c0
d	1a	27	34	41	4e	5b	68	75	82	8f	9c	a9	b6	c3	d0
e	1c	2a	38	46	54	62	70	7e	8c	9a	a8	b6	c4	d2	e0
f	1e	2d	3c	4b	5a	69	78	87	96	a5	b4	c3	d2	e1	f0
10	20	30	40	50	60	70	80	90	a0	b0	c0	d0	e0	f0	100

Base 16 (Hexadecimal)

Multiplication tables for bases 8 through 16. Multiples of the base r are circled. Product lines of divisors and semidivisors are shown in red and orange respectively. Yellow indicates the products of semitotatives. Light blue and light green indicate products of omega- and alpha-related totatives. The digit 2 in odd bases is both an alpha- and an omega-related totative. Light purple shows the products of 2 in an odd base. Light gray indicates product lines for "opaque" totatives in base r. In these diagrams the unit appears as an opaque totative, though its product line is trivial. See http://www.dozenal.org/articles/DSA-Mult.pdf for multiplication tables in bases 2 – 30, 32, 36, 40, and 60.

Appendix B: Digit Maps for Bases $2 \le r \le 60$

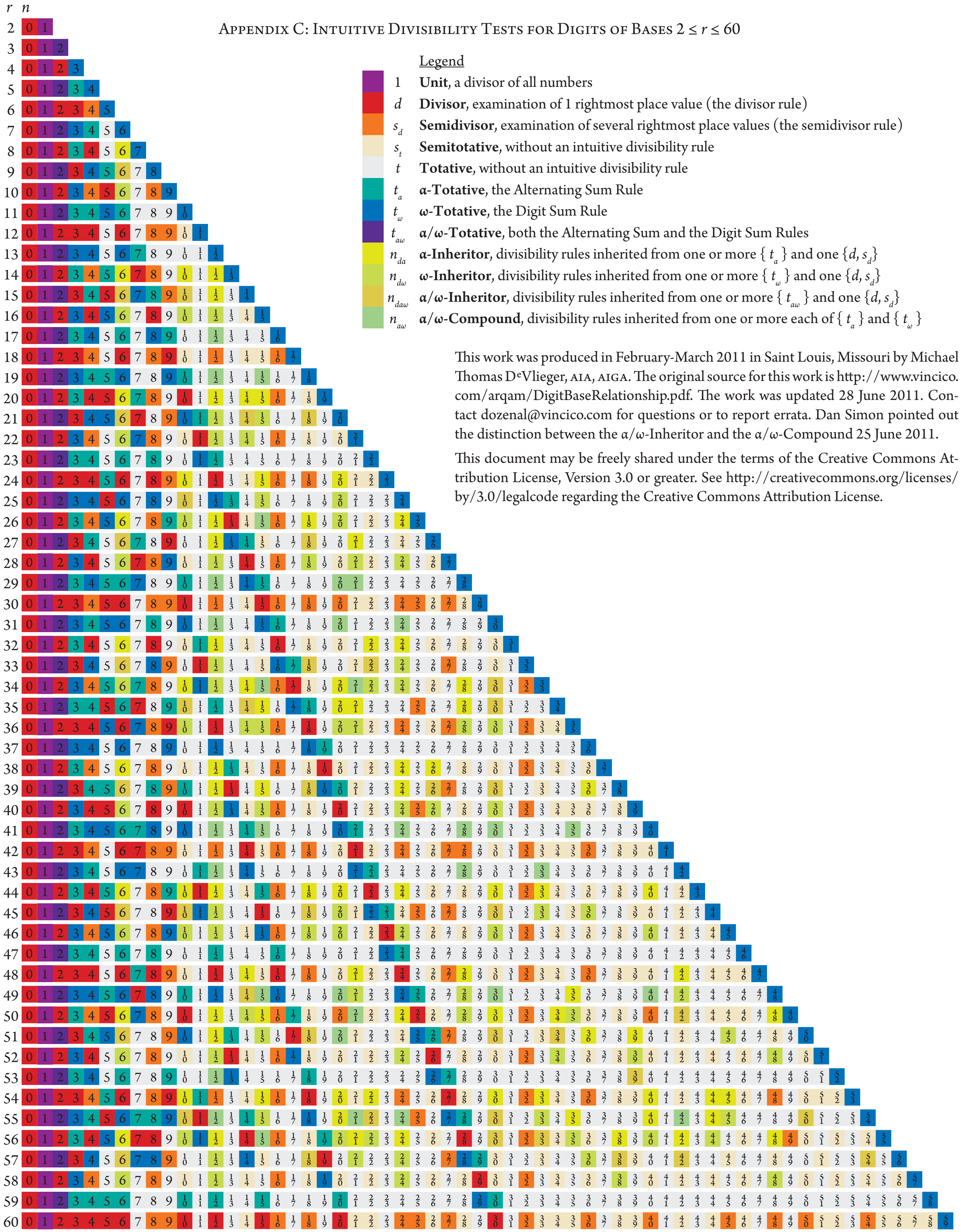


<u>Legend</u>

- 1 — **Unit**, $1 \mid r$ and $1 \perp r$
- d — **Divisor**, primes $d_p \mid r$, and composites $d_c \mid r$, the set of divisors of $r = \mathcal{D}_r$
- s_d — **Semidivisor**, a composite $s_d \mid kr$, where its divisors $d_s \in \mathcal{D}_r$
- s_t — **Semitotative**, a composite $s_t \mid kr$, where at least one of its divisors $d_s \notin \mathcal{D}_r$
- t — **Totative**, primes $t_p \perp r$, and composites $t_c \perp r$, the set of totatives of $r = \mathcal{T}_r$
- t_α — **α-Totative**, $t_\alpha \in \mathcal{T}_r$ and $t_\alpha \mid (r + 1)$
- t_ω — **ω-Totative**, $t_\omega \in \mathcal{T}_r$ and $t_\omega \mid (r - 1)$
- $t_{\alpha\omega}$ — α/ω-Totative, $t_{\alpha\omega} \in \mathcal{T}_r$, $t_{\alpha\omega} \mid (r + 1)$, and $t_{\alpha\omega} \mid (r - 1)$

See the following web pages for PDF versions of digit maps and digit spectra (http://www.vincico.com/arqam/digits/DigitBasePres.pdf), or an extension to base 120 (http://www.vincico.com/arqam/DigitBaseRelationship.pdf)

Appendix C: Intuitive Divisibility Tests for Digits of Bases $2 \le r \le 60$



Legend

- 1 — **Unit**, a divisor of all numbers
- d — **Divisor**, examination of 1 rightmost place value (the divisor rule)
- s_d — **Semidivisor**, examination of several rightmost place values (the semidivisor rule)
- s_t — **Semitotative**, without an intuitive divisibility rule
- t — **Totative**, without an intuitive divisibility rule
- t_α — **α-Totative**, the Alternating Sum Rule
- t_ω — **ω-Totative**, the Digit Sum Rule
- $t_{\alpha\omega}$ — **α/ω-Totative**, both the Alternating Sum and the Digit Sum Rules
- $n_{d\alpha}$ — **α-Inheritor**, divisibility rules inherited from one or more $\{ t_\alpha \}$ and one $\{d, s_d\}$
- $n_{d\omega}$ — **ω-Inheritor**, divisibility rules inherited from one or more $\{ t_\omega \}$ and one $\{d, s_d\}$
- $n_{d\alpha\omega}$ — **α/ω-Inheritor**, divisibility rules inherited from one or more $\{ t_{\alpha\omega} \}$ and one $\{d, s_d\}$
- $n_{\alpha\omega}$ — **α/ω-Compound**, divisibility rules inherited from one or more each of $\{ t_\alpha \}$ and $\{ t_\omega \}$

This work was produced in February-March 2011 in Saint Louis, Missouri by Michael Thomas De Vlieger, AIA, AIGA. The original source for this work is http://www.vincico.com/arqam/DigitBaseRelationship.pdf. The work was updated 28 June 2011. Contact dozenal@vincico.com for questions or to report errata. Dan Simon pointed out the distinction between the α/ω-Inheritor and the α/ω-Compound 25 June 2011.