Article Review: Mohr and Phillips 11↋↋_z (2015_d)

by John Volan

This document was adapted from the Article Reviews about Angular Units thread at the DozensOnline forum on tapatalk, in which I review journal articles supporting my assertion that angle should be treated as a first-class physical dimension, and that the coherent unit of angle can only be the radian, not the turn.

Dimensional Units in the SI, by Peter J. Mohr and William D. Phillips, *National Institute of Standards and Technology, Gaithersburg, MD 20899, USA* (11↋↋-05-24_z | 2015-05-28_d),[1] https://arxiv.org/pdf/1409.2794.

Quoting the abstract (my emphasis in bold red):

> **Mohr & Phillips 11↋↋_z (2015_d) wrote:**
>
> The International System of Units (SI) is supposed to be coherent. That is, when a combination of units is replaced by an equivalent unit, there is no additional numerical factor. Here we consider dimensionless units as defined in the SI, e.g. angular units like radians or steradians and counting units like radioactive decays or molecules. We show that an incoherence may arise when different units of this type are replaced by a single dimensionless unit, the unit "one", and suggest how to properly include such units into the SI in order to remove the incoherence. In particular, we argue that **the radian is the appropriate coherent unit for angles** and that **hertz is not a coherent unit in the SI**. We also discuss how including angular and counting units affects the fundamental constants.

Units and Dimensional Analysis

The article starts with a basic example of dimensional analysis: Calculate the kinetic energy E of a mass m moving with velocity v. Of course, it does this using SI units, but let's look at it in Primel units (brand abbreviation: ⊡) as well:

$$
\begin{aligned}
m &= 2\,\text{kg} \approx 2.12{\times}10^3{}_z\ \text{⊡ms}\ell = 2.12_z\ \text{⊡t↑ms}\ell = 2.12_z\ \text{⊡hd·ms} \\
v &= 3\,\text{m/s} = 10.8_d\,\text{km/h} \approx \text{↊}.7_z\ \text{⊡vc}\ell
\end{aligned}
\tag{0}
$$

In other words, the mass in question is 2 International System **kilograms**, or 2.12_z **prime·triqua·massels** (or **prime·hand·masses**). The velocity in question is 3 International System **meters per second**, which is equivalent, in non-coherent units, to 10.8_d ("ten decimal eight") **kilometers per hour**; this is about $\text{↊}.7_z$ ("ten zenimal seven") **prime·velocitels**. We want to know the kinetic energy in International System **joules**, as well as in **prime·energels**.

Here's the computation in SI, as specified in the article:

> **Mohr & Phillips 11↋↋_z (2015_d) wrote:**
>
> $$E = \frac{1}{2}mv^2 = \frac{1}{2}(2\text{ kg})(3\text{ m/s})^2 = \frac{2\cdot 3^2}{2}\text{kg m}^2\text{ s}^{-2} = 9\text{ J} \tag{1}$$

And here's the same computation in Primel:

$$
\begin{aligned}
E &= \frac{1}{2}mv^2 \approx \frac{1}{2}(2.12_z\ \text{⊡t↑ms}\ell)(\text{↊}.7_z\ \text{⊡vc}\ell)^2 = \frac{2.12_z\cdot 10_z^3\cdot \text{↊}.7_z^2}{2}\ \text{⊡ms}\ell\ \text{⊡vc}\ell^2 \\
&\approx 9.958{\times}10_z^4\ \text{⊡ng}\ell = 9.958_z\ \text{⊡q↑ng}\ell = 9.958_z\ \text{⊡hd·ng}
\end{aligned}
\tag{⊡1}
$$

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[1] This document annotates dozenal numbers with a subscript "z", and decimal numbers with a subscript "d". See https://dozenal.org/article-volan-base-annotation-schemes.

The computation may seem more complex in Primel, but it's not really. It's just that the original problem is biased in favor of simple, round, 1-digit numbers of SI coherent units, so the Primel equivalents for the same measurements necessarily require more digits of precision as well as scaling power prefixes. An example contrived to favor simple numbers in Primel would make the SI numbers look more complex. Plus, I'm taking the opportunity here to demonstrate some nice features of Primel.

First of all, note that all of the coherent units in Primel are named using **Quantitel** terminology (see my wiki page about that), qualified with the **prime·** brand prefix (abbreviated as brand mark ⊡). Quantitels are generic and reusable for any metrology, so in principle even SI could use them for its coherent units, qualified with a suitable brand prefix, perhaps **international·** (since it's literally the *International* System of Units), along with a suitable brand abbreviation, perhaps 🌐:

(🌐1) $$E = \frac{1}{2}mv^2 \approx \frac{1}{2}(2\,\text{🌐ms}\ell)(3\,\text{🌐vc}\ell)^2 = \frac{2\cdot 3^2}{2}\,\text{🌐ms}\ell\,\text{🌐vc}\ell^2 = 9\,\text{🌐ng}\ell$$

This would be under the assumption that:

- meter = **international·lengthel** (🌐lgℓ) = the SI coherent unit of *length*
- second = **international·timel** (🌐tmℓ) = the SI coherent unit of *time*
- kilogram = **international·massel** (🌐msℓ) = the SI coherent unit of *mass*
- meter per second = **international·velocitel** (🌐vcℓ) = the SI coherent unit of *velocity*
- joule = **international·energel** (🌐ngℓ) = the SI coherent unit of *energy*

But in addition to Quantitels, Primel provides **colloquial names** for specific scalings of its coherent units that turn out to be useful for everyday practical applications:

- The **prime·hand·length** (⊡hd·lg) is the same as the **prime·unqua·lengthel** (⊡u↑lgℓ) or $10^1{}_{\mathrm{z}}$ prime·lengthels, which approximates both a Metric decimeter as well as a 4-inch USC "hand" measure. The decimeter is designated by the ISO-2848 standard[2] as a "basic module" for the construction trades in Metric countries. The 4-inch USC "hand" is the comparable modular unit commonly used in the building trades in the US. (DozensOnline forum member "icarus", an architect by trade, can attest that *everything* in construction in the US comes in multiples of 4 inches.) In a Primel world, we can assume everything in construction would be in primel·hand·lengths.
- The **prime·hand·volume** (⊡hd·vm) is the same as the **prime·triqua·volumel** (⊡t↑vmℓ) or $10^3{}_{\mathrm{z}}$ prime·volumels. This is the cube of the prime·hand·length. It approximates a Metric cubic decimeter, or $10^3{}_{\mathrm{d}}$ cubic centimeters, also know as a liter.
- The **prime·hand·mass** (⊡hd·ms) is the same as the **prime·triqua·massel** (⊡t↑msℓ) or $10^3{}_{\mathrm{z}}$ prime·massels. This is the mass of a prime·hand·volume full of water at 1 prime·densitel. It approximates a kilogram, the mass of a liter, or $10^3{}_{\mathrm{d}}$ cubic centimeters, of water.
- The **prime·hand·weight** (⊡hd·wt), also known as the **prime·hand·force** (⊡hd·fc), is the same as the **prime·triqua·forcel** (⊡t↑fcℓ) or $10^3{}_{\mathrm{z}}$ prime·forcels. This is the weight of one prime·hand·mass under one **prime·accelerel** (⊡accℓ), Primel's standard value for Earth's gravity. This is approximately 10_{d} newtons or 1 decanewton, about the weight of a kilogram under SI's standard gravity. If the Metric meter had been assigned a value just $2\%_{\mathrm{d}}$ lower, 1 decanewton would have b een *exactly* the weight of a kilogagram under Earth's standard gravity.
- The **prime·hand·energy** (⊡hd·ng) is the same as the **prime·quadqua·energel** or $10^4{}_{\mathrm{z}}$ prime·energels. This is the work needed to lift a primel·hand·weight by one prime·hand·length. It approximates a joule, which is the work needed to lift a weight of 1 decanewton by 1 decimeter.

It should not be surprising, given the laws of physics, that if we scale the lengthel of any metrology by one order of magnitude in its preferred numeric base, we'd end up scaling its associated volume, and therefore its massel and forcel, by three orders of magnitude, but its energel by *four* orders of magnitude. That is, assuming that those coherent units were all derived from each other following the pattern of physical laws *without arbitrary fudge factors*.

Unfortunately, SI did *not* follow that pattern. The original Eighteenth-Century developers of Metric had *intended* the density of water to be the basis for deriving the gram from a cubic centimeter. Indeed under the Nineteenth-Century's coherent Centimeter-Gram-Second (CGS) system, the gram, the cubic centimeter, and the density of water did become the respective massel, volumel, and densitel of CGS. But under the Twentieth-Century's coherent International System of Units, the density of water is *not* the SI

[2] https://en.wikipedia.org/wiki/ISO_2848

densitel. Instead, there is a fudge factor that sets the SI densitel 1000$_d$ times lower. That's just so that the kilogram (or "grave") could get to be SI's massel, when by right, it should have been the megagram (Mg) or Metric "tonne" (t), which is the mass of a *cubic meter* of water — because SI's actual volumel is the cubic meter, *not* the liter.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> This calculation illustrates the important principle of the SI that units are coherent. That is, when a combination of units is replaced by an equivalent unit, there is no additional numerical factor. For Eq. (1) this corresponds to the relation
>
> (2) $\quad \mathrm{kg\,m^2\,s^{-1}} = \mathrm{J}$

The equivalent in Primel would be:

(⊡2) ⊡msℓ ⊡lgℓ^{2} ⊡tmℓ$^{-2}$ = ⊡ngℓ

But in this example, Primel was able to express a slightly different unit relationship:

(⊡2b) ⊡msℓ ⊡vcℓ^{2} = ⊡ngℓ

In other words, one prime·massel (⊡msℓ) times one prime·velocitel squared (⊡vcℓ^{2}) can immediately be equated to one prime·energel (⊡ngℓ), without needing to expand out the prime·velocitel into one prime·lengthel per prime·timel (⊡lgℓ ⊡tmℓ$^{-1}$). We don't need to see the fully expanded product ⊡msℓ ⊡lgℓ^{2} ⊡tmℓ$^{-2}$ in order to realize we have an energy unit. That is a step that can be skipped with knowledge of a physical law — which the *problem itself is already quoting*:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> (1) $\quad E = \frac{1}{2} m v^2 = \cdots$

But SI has *no direct name* for most of its coherent unit, including its coherent unit of velocity. Instead it punts with the ridiculous notion that a *formula* like "meter per second" can be deemed a "derived unit name". But this can be cured by applying Quantitels:

(🌐2b) 🌐msℓ 🌐vcℓ^{2} = 🌐ngℓ

So an international·massel times an international·velocitel squared equals an international·energel. By provisioning *every* type of quantity with a sensible and transparent name for its coherent unit, based directly on the *name of the quantity type itself*, Quantitels make the distinction between so-called "base units" and "derived units" a completely arbitrary one. As long as the *relationships* between Quantitels follow the pattern of physical law, without introducing fudge factors, then it is *unnecessary* to always break down a dimensional analysis of a unit into a formula containing only some arbitrary subset of "base" units.

For instance, should the **ampere** (the **international·currentel**, SI's coherent unit of *current*) be a "base" unit, as SI designates it? Should the **coulomb** (the **international·electrel**, SI's coherent unit of static *electric* charge) be a "derived" unit? Static electric charge seems to be the unique primitive dimension underlying the science of electromagnetism, the novel phenomenon over and above the phenomena of time, length, and mass underlying classical Newtonian mechanics. Whereas current seems more of a derivative property, the rate of flow of electric charge with respect to time. But Primel *does not care* which dimensions are "primitive" and which are "derived". As long as you can say that one **currentel** equals one **electrel** per **timel**:

(🌐2c) $\text{🌐ctℓ} = \dfrac{\text{🌐εtℓ}}{\text{🌐tmℓ}}$

or that one **electrel** equals one **currentel** times one **timel**:

(🌐2d) 🌐εtℓ = 🌐ctℓ · 🌐tmℓ

then it *does not matter either way*.

That said, if it *helps* to do a full dimensional breakdown of some quantity into a formula containing only "primitive" dimensions, to reassure yourself that two quantities are really of the same type, there's no problem doing that. But if you understand natural laws well enough (and assuming those laws are well-formed and dimensionally balanced), then it shouldn't always be *necessary* to do that.

The article then introduces the notation $q = \{q\}\,[q]$ to show that any dimensioned quantity q can be broken down into a dimensionless coefficient $\{q\}$ and a unit $[q]$ that carries the dimensionality. The article then use that notation to rewrite equation (1):

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
> (3) $\{E\}\,[E] = \frac{1}{2}\,\{m\}\,[m]\,(\{v\}\,[v])^2$

Except that this really should have been qualified to indicate that the breakdown is specifically in terms of SI coherent units:

(3a) $$\{E\}_{🌐}\,[E]_{🌐} = \frac{1}{2}\,\{m\}_{🌐}\,[m]_{🌐}\,(\{v\}_{🌐}\,[v]_{🌐})^2$$

Although the article doesn't explicitly show it, it implies that the following transformation could be done, to gather all the dimensionless coefficients together, and all the units together:

(3b) $$\{E\}_{🌐}\,[E]_{🌐} = \left(\frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\right)\left([m]_{🌐}\,[v]_{🌐}{}^2\right)$$

It then particularizes equation (3b) for SI coherent units:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
> (4) $\{E\}\ \mathrm{J} = \left(\frac{1}{2}\,\{m\}\,\{v\}^2\right)\left(\mathrm{kg\,m^2\,s^{-1}}\right) = \frac{1}{2}\,\{m\}\,\{v\}^2\ \mathrm{J}$

Well, it *sort of* particularized (3b). It actually skipped a step:

(4b) $$\{E\}_{🌐}\ \mathrm{J} = \left(\frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\right)\left(\mathrm{kg}\ (\mathrm{m/s})^2\right) = \left(\frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\right)\left(\mathrm{kg\,m^2\,s^{-1}}\right) = \frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\ \mathrm{J}$$

In (4) it jumped straight to expanding out $[v]_{🌐}{}^2$. But once again, Primel can more directly represent $[m]_{\boxdot}\,[v]_{\boxdot}{}^2$ and jump from that straight to $[E]_{\boxdot}$ without ever needing to expand out $[v]_{\boxdot}{}^2$:

($\boxdot$4) $$\{E\}_{\boxdot}\ \boxdot\mathrm{ng}\ell = \left(\frac{1}{2}\,\{m\}_{\boxdot}\,\{v\}_{\boxdot}{}^2\right)\left(\boxdot\mathrm{ms}\ell\ \boxdot\mathrm{vc}\ell^2\right) = \frac{1}{2}\,\{m\}_{\boxdot}\,\{v\}_{\boxdot}{}^2\ \boxdot\mathrm{ng}\ell$$

With Quantitels, this could even be done in SI:

(🌐4) $$\{E\}_{🌐}\ 🌐\mathrm{ng}\ell = \left(\frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\right)\left(🌐\mathrm{ms}\ell\ 🌐\mathrm{vc}\ell^2\right) = \frac{1}{2}\,\{m\}_{🌐}\,\{v\}_{🌐}{}^2\ 🌐\mathrm{ng}\ell$$

Angles

In the next section, Mohr and Phillips consider angles as dimensioned quantities:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> In part because units are rarely considered in mathematics, the unit of radian for angles is rarely mentioned in the mathematics reference literature, just as the meter is also rarely mentioned. Units are unnecessary in purely mathematical analysis. By the same token, caution is necessary in drawing conclusions about units based on purely mathematical considerations. For example, in the current SI, it is stated that angles are dimensionless based on the definition that an angle in radians is arc length divided by radius, so the unit is surmised to be a derived unit of one, or a dimensionless unit.

Apparently they were thinking of this as the defining equation for an angle:

(A) $\quad \theta = \dfrac{s}{r}$

where:

s = **arc length** on a circle, subtending angle θ
Dimension: length

r = **radius** of the circle
Dimension: length

θ = **angle**
Dimension: length/length = 1 (dimensionless) **?**

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> However, this reasoning is not valid, as indicated by the following example. An angle can also be defined as "twice the area of the sector which the angle cuts off from a unit circle whose centre is at the vertex of the angle."[7] This gives the same result for the numerical value of the angle as the definition quoted in the SI Brochure, however by following similar reasoning, it suggests that angles have the dimension of length squared rather than being dimensionless. This illustrates that conclusions about the dimensions of quantities based on such reasoning are clearly nonsense.

Although they didn't provide an equation, apparently they were thinking of this:

(B) $\quad A_\angle = \dfrac{1}{2}r^2\theta \quad \rightarrow \quad \theta = \dfrac{2A_\angle}{r^2}$

where:

$A_\angle$ = **area of a sector** of a disc covering the subtended angle θ
Dimension: area = length^2

r = **radius** of the disc
Dimension: length

r^2 = square **radius**
Dimension: length^2 = area

θ = the subtended **angle**
Dimension: area/area = 1 (dimensionless) **?**

I think Mohr & Phillips were mistaken that Eq. (B) would confer the dimension of area on θ. It clearly would have made it just as dimensionless as Eq. (A) would have. And since they admit that Eq. (B) would give θ the same numeric value as Eq. (A), I think it's a wash. However, both equations suffer from failing to confer true dimensionality on θ, which means there's something wrong with the dimensionality of one of the other variables in both equations.

As I argued in the Why did the SI choose h instead of h-bar? thread on the DozensOnline forum on Tapatalk, what we really need to do is leave the status-quo as-is, where “angles” are treated as dimensionless and radian-specific, where existing “angle” units are all dimensionless numbers, and where the “radian” itself is equal to the dimensionless number 1. Instead, we should introduce a concept of a “true·angle” dimension, with **true·radian** (ⓣrad) as the coherent unit, and use diacritics to mark whenever symbols have been dimensionally corrected by either being multiplied by a true·radian:

(C) $\quad \overset{\frown}{1} = 1 \times \text{ⓣrad} = \text{ⓣrad}$

or divided by a true·radian:

(D) $\quad \overset{\frown}{1} = \dfrac{1}{\text{ⓣrad}} = \text{ⓣrad}^{-1} = \text{ⓣrad}\backslash$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> The angle θ is thus given by
>
> (5) $\quad \theta = \dfrac{s}{r}\,\text{rad}$
>
> which corresponds to $\{\theta\} = s/r$ and $[\theta] = \text{rad}$

Let’s recast this as generating a new symbols $\overset{\frown}{\theta}$ for the true·angle, defined as the dimensional correction of the dimensionless θ by multiplying by a true·radian. In which case, its unitless coefficient is simply θ itself:

(E) $\quad \{\overset{\frown}{\theta}\} = \theta$

and its units are of course true·radians:

(F) $\quad [\overset{\frown}{\theta}] = \text{ⓣrad}$

Putting these together gives us a true·angle:

(G) $\quad \overset{\frown}{\theta} = \{\overset{\frown}{\theta}\}[\overset{\frown}{\theta}] = \theta\ \text{ⓣrad}$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> In view of the problems that can occur if units for angles are omitted, we consider the consequences of a consistent treatment of units for angles in the following. For an infinitesimal segment of a plane curve, the change in angle $\mathrm{d}\theta$ of the tangent to the curve is proportional to the infinitesimal change in position $\mathrm{d}s$ along the curve,
>
> (8) $\quad \mathrm{d}\theta = \mathcal{C}\mathrm{d}s$
>
> where $\mathcal{C}$ has units of rad/m. Evidently, the angular curvature is a measure of the amount of bending of the segment of the curve.

They’re using a different symbol (an uppercase calligraphic $\mathcal{C}$), but this is exactly the quantity of curvature ($\varkappa$) which many have described. I’d just annotate this, to correct its dimensionality:

$\overset{\frown}{\varkappa}$ = curvature Dimension: true·angle / length

(8b) $\quad \mathrm{d}\overset{\frown}{\theta} = \overset{\frown}{\varkappa}\mathrm{d}s$

This means an infinitesimal change in true·angle equals the curvature of the circle (a constant) times the infinitesimal change in arc length. But this can be rearranged to define curvature as the differential of true·angle with respect to arc length:

(8c) $$\overset{\frown}{\kappa} = \frac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}s}$$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> if the angular radius of curvature $\mathcal{R}$ is defined as
>
> (9) $$\mathcal{R} = \frac{1}{\mathcal{C}}$$

Mohr & Phillips recognize the reciprocal of curvature, and use the conventional term for it, "radius of curvature". The use a different symbol (uppercase calligraphic $\mathcal{R}$), but this is precisely the quantity I call "radiality" ($\underset{\smile}{r}$). I bse the symbol for that on the symbol for the radius of the circle r, but dimensionally corrected to divide it by a true·radian:

r = radius
Dimesion: length

$\underset{\smile}{r}$ = radiality
Dimension: length / true·angle

(9a) $$\underset{\smile}{r} = \frac{r}{\text{ⓣrad}}$$

(9b) $$\underset{\smile}{r} = \frac{1}{\overset{\frown}{\kappa}}$$

But even Mohr & Phillip's calligraphic $\mathcal{R}$ effectively ties back to r, just indicating a different dimensionality by means of a stylistic difference, rather than by a diacritic annotation.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> then
>
> (10) $$\mathrm{d}\theta = \frac{\mathrm{d}s}{\mathcal{R}}$$

So the infinitesimal change in true·angle equals the infinitesimal change in arc-length divided by the radiality of the circle:

(10a) $$\mathrm{d}\overset{\frown}{\theta} = \frac{\mathrm{d}s}{\underset{\smile}{r}}$$

This can be rearranged to define radiality as the instantaneous rate of change of arc-length with respect to true·angle:

(10b) $$\underset{\smile}{r} = \frac{\mathrm{d}s}{\mathrm{d}\overset{\frown}{\theta}}$$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> The quantity $\mathcal{R}$ with units m/rad should be distinguished from r in Eq. (5) which has units of m. If the curve is a portion of a circle, then we have
>
> (11) $$\theta = \frac{s}{\mathcal{R}}$$
>
> in analogy with Eq. (5) …

So the true·angle itself is its subtending arc-length divided by the radiality of the circle:

(11a) $$\overset{\frown}{\theta} = \frac{s}{\underset{\smile}{r}}$$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> … and $\{\mathcal{R}\} = \{r\}$.

In other words, radius and radiality have exactly the same dimensionless coefficient, the only difference is in their units (and therefore their dimensionality):

(11b) $$\{\underset{\smile}{r}\} = \{r\}$$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> In fact, Eq. (5) is the same as Eq. (11) if the replacement rad → 1 is made.

But that just gives us Eq. (A) that we saw before:

(A) $$\theta = \frac{s}{r}$$

where unadorned θ is kept dimensionless, in keeping with the status quo.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> This extends naturally to steradians for solid angle, abbreviated sr, for which an infinitesimal solid angle subtended by the area da on the surface of a sphere is given by
>
> (12) $$\mathrm{d}\Omega = \frac{\mathrm{d}a}{\mathcal{R}^2}$$
>
> which has units of rad^2.

This agrees with how I handle solid angles:

(12a) $$\mathrm{d}\overset{\frown}{\Omega} = \frac{\mathrm{d}A}{\underset{\smile}{r}^2}$$

Dimension: true·solid·angle = true·angle^2
Unit: true·steradian = true·radian^2 (ⓣsr = ⓣrad^2)

Mohr & Phillips summarize their treatment of angles in a table:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> **TABLE I: Quantities involving angles and their units.**
>
> | QUANTITY | EQUATION | UNITS |
> |---|---|---|
> | angle | θ | rad |
> | angular curvature | $\mathcal{C} = \frac{d\theta}{ds}$ | rad m^{-1} |
> | angular radius of curvature | $\mathcal{R} = \frac{1}{\mathcal{C}} = \frac{ds}{d\theta}$ | m rad^{-1} |
> | infinitesimal arc length | $ds = \mathcal{R}\, d\theta$ | m |
> | infinitesimal angle | $d\theta = \frac{ds}{\mathcal{R}}$ | rad |
> | solid angle | Ω | sr |
> | infinitesimal surface area | da | m^2 |
> | infinitesimal solid angle | $d\Omega = \frac{da}{\mathcal{R}^2}$ | sr = rad^2 |

But this table can be reformulated in terms of true·angles, with diacritic annotations to indicate the needed dimensional repairs. Moreover we can generalize it to any coherent system of units, by using Quantitels:

Table Ib: Quantitities involving true·angles and their units.

QUANTITY	EQUATION	UNITS
angle	$\overset{\frown}{\theta}$	ⓣrad
curvature	$\overset{\frown}{\kappa} = \dfrac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}s}$	$\mathrm{cv}\ell = $ ⓣrad $\mathrm{lg}\ell^{-1}$
angular radius of curvature	$\underset{\smile}{r} = \dfrac{1}{\overset{\frown}{\kappa}} = \dfrac{\mathrm{d}s}{\mathrm{d}\overset{\frown}{\theta}}$	$\mathrm{rd}\ell = \mathrm{lg}\ell$ ⓣrad^{-1}
infinitesimal arc length	$\mathrm{d}s = \underset{\smile}{r}\,\mathrm{d}\overset{\frown}{\theta}$	$\mathrm{lg}\ell$
infinitesimal angle	$\mathrm{d}\overset{\frown}{\theta} = \dfrac{\mathrm{d}s}{\underset{\smile}{r}}$	ⓣrad
solid angle	$\overset{\frown\frown}{\Omega}$	ⓣsr
infinitesimal surface area	$\mathrm{d}a$	$\mathrm{ar}\ell = \mathrm{lg}\ell^2$
infinitesimal solid angle	$\mathrm{d}\overset{\frown\frown}{\Omega} = \dfrac{\mathrm{d}a}{\underset{\smile}{r}^2}$	ⓣsr $=$ ⓣrad^2

Next, the article moves on to exponential and trigonometric functions, which are so central to angular measurement:

> **Mohr & Phillips 11↋↋$_z$ (2015$_d$) wrote:**
>
> In applications, angles appear in the exponential and trigonometric functions, and these functions are defined for an argument that is a dimensionless number, i.e., the numerical value of the angle expressed in radians. The exponential function is given by its power series
>
> (13) $$e^x = 1 + x + \frac{x^2}{2} + \cdots$$
>
> and then the relation
>
> (14) $$e^{iy} = \cos y + i \sin y$$
>
> follows from the series expansions of the cosine and sine functions. The unit "radian" cannot be included as a factor in the arguments of these functions, because every term in the power series must have the same unit.

Except that if "radian" is just considered a dimensionless number equivalent to the pure number 1 anyway, then all of its powers would also be the dimensionless number 1, and then none of them would have any dimensional impact on the terms in the power series. They'd all stay commensurate with each other. It's better to distinguish a true·angle dimension, with a true·radian unit, but then make sure this never gets into these transcendental exponential and trigonometric functions, which require their argument to be dimensionless.

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> The connection of these functions to angles follows from the fact that Eq. (14) is a point in the complex plane on the unit circle at an angle $\theta = y$ rad in the counter-clockwise direction from the positive real axis. The periodicity of the function e^{iy} fixes the unit of the angle to be $[\theta] =$ rad, because both the angle $\theta = y$ rad and the function e^{iy} go through one complete cycle as $y = \{\theta\}$ ranges from 0 to 2π.

Mohr and Phillips's discourse here could benefit from the simple expedient of defining:

(τ) $\quad \tau = 2\pi$

Well, this paper was published only 5 years after Michael Hartl wrote "The Tau Manifesto".[3] Perhaps it hadn't trickled out to them yet. Or perhaps they were just too πous to become τists.

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> **The choice of any other unit for [θ] would not align these two periods.**
>
> However, it is the general practice in physics to write the exponential function of an angle $\theta = y$ rad as $e^{i\theta}$ rather than e^{iy} or $e^{i\{\theta\}}$. In fact, in carrying out calculations, scientists do not usually distinguish between θ and $\{\theta\}$, **which amounts to treating** rad **as being 1.**
>
> This reveals a conflict between consistent application of dimensional analysis and common usage.

Needless to say, my stance is to just not fight the "common usag" and leave as-is the status quo interpretation of "angles" as dimensionless, but strongly mark where we are introducing true·angle dimension. So rather than distinguish θ as having dimension and $\{\theta\}$ as dimensionless, it's better to leave θ dimensionless and mark $\overset{\frown}{theta}$ as having true dimension. Likewise, leave rad $= 1$ but mark ⓣrad as having true dimension. Let the standard transcendental functions assume their arguments are strictly dimensionless, so that expressions like

(13b) $\quad e^{\theta} = 1 + \theta + \dfrac{\theta^2}{2!} + \dfrac{\theta^3}{3!} + \cdots$

can remain just as they are. But then introduce "complete" versions of the transcendental functions, strongly marked as such, that can take dimensioned arguments, but have them extract the dimensionless coefficient from their argument and pass that on to the standard version of the function:

(13c) $\quad \underset{\smile}{\exp}(\overset{\frown}{\theta}) = e^{\underset{\smile}{\overset{\frown}{\theta}}} = e^{\theta} = \cdots$

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> …for general use in printed equations following the common practice, the argument of the exponential and trigonometric functions is simply written as θ which corresponds to replacing rad by 1. Of course, this replacement can only be done for the unit rad, and not revolutions or degrees, replacements that would introduce numerical factors. In this sense, the unit rad is a coherent unit in the SI, whereas revolutions and degrees are not.

As I argued throughout Why did the SI choose h instead of h-bar?, the same logic would apply to any metrology that aspires to be a "coherent system of units", leading to my conclusion that the radian is the only possible coherent unit of angle.

[3] https://www.tauday.com/

Periodic Phenomena

In the next section, Mohr and Phillips challenge whether the hertz (Hz) really should be considered a coherent unit under SI — at least, as a coherent unit of rotational frequency.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> Periodic phenomena in physics include rotations of an object, cycles or repetitions of a wave, or a series of any regular, repetitive events. Such periodic phenomena are characterized by a frequency whose units can be an angular factor or a cycle divided by time. In the SI, cycles/second = cyl/s is named hertz or Hz, and
>
> (15) $\quad 1\ \mathrm{Hz} = 1\ \mathrm{cyl\ s^{-1}} = 2\pi\ \mathrm{rad\ s^{-1}}$
>
> where the second equality follows from Eq. (7) with revolution replaced by cycle. Hz maybe viewed as being equivalent to rotations per second, but often, “rotations” is used for mechanical motion and “cycles” is used for waves.

Honestly, I always thought that hertz could be interpreted both as “waves per second” and “turns per second”, because SI treats both “waves” and “turns” as equally dimensionless. But a true·turn (ⓣtr) unit is already part of my proposal for true·angle units with true dimensionality. I think true·wave (ⓣwv) can be a synonym for true·turn because they seem to share the same dimensionality, just different applications. (This would be analogous to how workel (wkℓ) (generic coherent unit of work) and heatel (htℓ) (generic coherent unit of heat) would both just be synonyms for energel (ngℓ) (generic coherent unit of energy), because “work” ” and “heat” are just two different types of “energy”.)

Notice their mention of “a series of any regular, repetitive events.” I think frequency of repetitions is a distinct concept (and therefore a distinct dimension) from frequency of rotation. Yet hertz seems to have been applied to that as well. Mohr and Phillips actually tackle that in the next section, so let’s table that for now.

But let me assert this: Whereas the International System hertz unit seems to come with extra baggage, in contrast Primel (and other similar metrologies that use Quantitels) define “frequency” or “rate” as simply the reciprocal of “time” — and nothing else. The **frequenciel**, or **frequel** for short (fqℓ), or synonymously, the **ratel** (rtℓ), is defined as the reciprocal of the **timel** (tmℓ), whatever that happens to be in a particular metrology. It isn’t the “frequency” or “rate” of anything in particular, not even implicitly. It’s not even implicitly a frequency of rotation. To measure a frequency or rate of some specific phenomenon, the **frequel** or **ratel** must be combined with something that carries the dimension of that phenomenon.

For instance, if you specifically want the frequency or rate of rotation, you’d need to measure that as “true·radian frequency” using the unit **true·radian · frequel** (ⓣrad·fqℓ). But in that case, that would simply be a synonym for “angular·velocity” (symbolized $\widehat{\omega}$), measured in **ang·velocitels** (∡vcℓ). You could also measure it as “true·turn frequency” using the unit **true·turn · frequel** (ⓣtr·fqℓ), but that would be a non-coherent unit.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> The symbol used for angular frequency is ω, which is understood to mean the frequency in units of rad/s, while the symbols ν or f are used to denote frequency expressed in hertz. The relation between the numerical value of a particular frequency expressed in Hz or rad s^{-1} is given by
>
> (16) $\quad \{\nu\}_{\mathrm{Hz}}\,[\mathrm{Hz}] = \{\omega\}_{\mathrm{rad\cdot s^{-1}}} = \{\omega\}_{\mathrm{rad\cdot s^{-1}}}\,[\mathrm{rad\cdot s^{-1}}] = \dfrac{\{\omega\}_{\mathrm{rad\cdot s^{-1}}}}{2\pi}\,[\mathrm{Hz}]$
>
> or
>
> (17) $\quad \dfrac{\{\omega\}_{\mathrm{rad\cdot s^{-1}}}}{2\pi} = \{\nu\}_{\mathrm{Hz}}$
>
> where the second equality in Eq. (16) follows from Eq. (15).

I would recast this in terms of true·angle units as well as generic Quantitels:

(16a) $\{\overset{\odot}{\nu}\}_{\text{ⓣtr·fqℓ}}\ \text{ⓣtr·fqℓ} = \{\overset{\frown}{\omega}\}_{\text{ⓣrad·fqℓ}}\ \text{ⓣrad·fqℓ}$

and then note that

(16b) $\text{ⓣtr} = \tau\ \text{ⓣrad}$

so that

(16b) $\tau\ \{\overset{\odot}{\nu}\}_{\text{ⓣtr·fqℓ}}\ \text{ⓣrad·fqℓ} = \{\overset{\frown}{\omega}\}_{\text{ⓣrad·fqℓ}}\ \text{ⓣrad·fqℓ}$

and therefore

(17a) $\tau\ \{\overset{\odot}{\nu}\}_{\text{ⓣtr·fqℓ}} = \{\overset{\frown}{\omega}\}_{\text{ⓣrad·fqℓ}}$

or

(17b) $\{\overset{\odot}{\nu}\}_{\text{ⓣtr·fqℓ}} = \dfrac{\{\overset{\frown}{\omega}\}_{\text{ⓣrad·fqℓ}}}{\tau}$

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> As already noted, radians behave as coherent units for the SI, so we make the identification $\{\omega\}_{\mathrm{rad \cdot s^{-1}}} = \{\omega\}$, where the curly brackets with no subscript indicate that the numerical value corresponds to a coherent SI unit.

Rather than just reflexively assume SI, here I would do a brand mark instead: $\{\overset{\frown}{\omega}\}_{\text{🌐}}$.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> However, a consequence of this convention is that the unit Hz is not a coherent SI unit as indicated by Eq. (17). This is in conflict with the current SI where Hz is treated as a coherent SI unit, only because cyl is replaced by "one". Since this leads to an inconsistency, we propose that the SI be modified in such a way that Hz is neither treated as a coherent SI unit nor replaced by $\mathrm{s^{-1}}$.

I wholeheartedly agree with Mohr and Phillips: The turn is not the coherent unit of angle. The radian is. So the turn per timel is not a coherent unit either.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> A basic equation for waves is the relation between the wavelength and the frequency. This is generally written
>
> (19) $\lambda\nu = c$
>
> where λ is the crest to crest wavelength and c is the wave velocity, which for electromagnetic radiation in free space is the speed of light. From the requirement that units on both sides of an equality must be the same, and the conventions that c has the unit m/s in the SI and ν has the unit Hz, Eq. (19) implies that the unit for λ is
>
> (20) $[\lambda] = \dfrac{[c]}{[\nu]} = \mathrm{m\,s^{-1}\,Hz^{-1}} = \mathrm{m\,cyl^{-1}}$
>
> which has a self-evident intuitive interpretation.

I would express that in generic Quantitels and true·angle units as **lengthels per true·turn** (lgℓ ⓣtr^{-1}). But this would not be a coherent unit.

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> Neither Hz nor m/cyl is a coherent unit. For a "coherent" version of Eq. (19), that is, an equation in which c has the unit m/s and the frequency has the unit rad/s, we write
>
> (21) $\quad \bar{\lambda}\omega = c$
>
> which implies that the reduced wavelength $\bar{\lambda}$ has the units
>
> (22) $\quad \bar{\lambda} = \dfrac{[c]}{[\omega]} = \dfrac{\mathrm{m\,s^{-1}}}{\mathrm{rad\,s^{-1}}} = \mathrm{m\,rad^{-1}}$

Where the **reduced wavelength** is $\bar{\lambda} = \frac{\lambda}{\tau}$. I would correct and annotate the dimensionality of Eq. (21) as:

(21a) $\quad \underset{\smile}{\bar{\lambda}}\overset{\frown}{\omega} = c$

and describe the units of $\underset{\smile}{\bar{\lambda}}$ as **lengthels per true·radian** (lgℓ ⓣrad^{-1}). This is the same as the **radiel** (rdℓ), the (generic) coherent unit of radilaity, as in $\underset{\smile}{r}$. Perhaps we can change the term "reduced wavelength" to *wave radiality*, i.e., how far a waveform travels (in lengthels), per radian of "waving".

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> Another quantity associated with waves is the wave vector
>
> (25) $\quad k = \dfrac{1}{\bar{\lambda}}$
>
> it has units of radians per meter or rad/m.

I would correct and annotate the dimensionality of Eq. (25) as:

(25a) $\quad \overset{\frown}{k} = \dfrac{1}{\underset{\smile}{\bar{\lambda}}}$

and describe its units generically as **true·radians per lengthel** (ⓣrad·lgℓ$^{-1}$), which is the same as the **curvel** (cvℓ), the (generic) coherent unit of *curvature*, just like $\overset{\frown}{\kappa}$. Perhaps we could change the term for "wave vector" or "wave number" to *wave curvature*, i.e., how much "waving" a waveform does (in true·radians), per lengthel of travel.

Mohr and Phillips summarize the units for angular mechanics in a table:

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> **TABLE II. Quantites involving rotational motion and their units.**

QUANTITY	EQUATION	UNITS
angular velocity	$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$	rad s^{-1}
angular acceleration	$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$	rad s^{-2}
velocity	$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \mathcal{R}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \mathcal{R}\omega$	m s^{-1}
moment of inertia	$I = m\mathcal{R}^2$	kg m^2 rad^{-2}
angular momentum	$L = I\omega = m\mathcal{R}^2\omega$	J s rad^{-1}
torque	$N = I\alpha$	J rad^{-1}
energy	$E = I\frac{\omega^2}{2}$	J
centrifugal force	$F_{\mathrm{c}} = m\omega^2\mathcal{R} = \frac{mv^2}{\mathcal{R}}$	N rad

I would recast this in terms of generic Quantitels and use diacritics to annotate dimensional corrections in the symbols:

TABLE IIb. Quantities involving rotational motion and their units.

Quantity	Equation	Units
angular velocity	$\overset{\frown}{\omega} = \dfrac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}t}$	∡vcℓ= cvℓ vcℓ= ⓣrad tmℓ$^{-1}$
angular acceleration	$\overset{\frown}{\alpha} = \dfrac{\mathrm{d}\overset{\frown}{\omega}}{\mathrm{d}t} = \dfrac{\mathrm{d}^2\overset{\frown}{\theta}}{\mathrm{d}t^2}$	∡accℓ= cvℓ accℓ= ⓣrad tmℓ$^{-1}$
velocity	$v = \dfrac{\mathrm{d}s}{\mathrm{d}t} = \underset{\smile}{r}\dfrac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}t} = \underset{\smile}{r}\overset{\frown}{\omega}$	vcℓ
anrgular mass (moment of inertia)	$\underset{\smile}{\underset{\smile}{I}} = m\underset{\smile}{r}^2$	∡msℓ= msℓ rdℓ^2 = msℓ lgℓ^2 ⓣrad^{-2}
angular momentum	$\underset{\smile}{L} = \underset{\smile}{\underset{\smile}{I}}\overset{\frown}{\omega} = m\underset{\smile}{r}^2\overset{\frown}{\omega}$	∡mmℓ= ∡msℓ ∡vcℓ= rdℓ mmℓ= actℓ ⓣrad^{-1}
torque	$\underset{\smile}{T} = \underset{\smile}{\underset{\smile}{I}}\overset{\frown}{\alpha}$	∡fcℓ= ∡msℓ ∡accℓ= rdℓ fcℓ= ngℓ ⓣrad^{-1}
energy	$E = \underset{\smile}{\underset{\smile}{I}}\dfrac{\overset{\frown}{\omega}^2}{2}$	ngℓ= ∡msℓ ∡vcℓ^2
centrifugal force	$F_{\mathrm{c}} = m\overset{\frown}{\omega}^2\underset{\smile}{r}\cdot\underset{\smile}{1} = \dfrac{mv^2}{\underset{\smile}{r}}\cdot\underset{\smile}{1} = \dfrac{mv^2}{r}$	fcℓ

where:

- vcℓ= **velocitel** = coherent unit of *velocity*
- mmℓ= **momentumel** = coherent unit of *momentum*
- actℓ= **actionel** = coherent unit of *action*
- fcℓ= **forcel** = coherent unit of *force*
- ngℓ= **energel** = coherent unit of *energy*
- ∡vcℓ= **ang· velocitel** = coherent unit of *angular velocity*
- ∡accℓ= **ang·accelerel** = coherent unit of *angular acceleration*
- ∡msℓ= **ang·massel** = coherent unit of *angular mass*, or *moment of inertia*
- ∡mmℓ= **ang·momentumel** = coherent unit of *angular momentum*
- ∡fcℓ= **ang·forcel** = coherent unit of *angular force*, or *torque*

Overall, Mohr and Phillips corroborate my approach to granting true·angular dimension to angles.

However, I think they made an error with centrifugal force: It should be simply a force without a stray angle dimension in the denominator. The derivation of the formula for F_{c} using calculus should have generated an extra radianic factor $(\underset{\smile}{1})$ to cancel this out.

Counting Quanities

In the next section, Mohr and Phillips consider physical quantities that involve counting discrete entities, such as molecules or atoms of a substance, or discrete events, such radioactive decays or counts of such in a detector. Such quantities have been assumed to be dimensionless under SI, but they question that assumption.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> For example, if in a certain time interval there are $\mathcal{D}$ = 200 decays = 200 dcy and the detector registers $\mathcal{N}$ = 20 counts = 20 cnt, then the efficiency η of detection is
>
> (30) $$\eta = \frac{\mathcal{N}}{\mathcal{D}} = 0.1\frac{\text{cnt}}{\text{dcy}}$$

I could see introducing a distinct dimension for radioactive decay event counting, with a Quantitel of **radioactivel** (rdctvℓ) representing one radioactive decay event. There could even be another completely distinct dimension for detection event counting in scientific instruments, with Quantitel **detectionel** (dtctℓ) representing one detection event. Then the ratio of detectionels to actual radioactivels would represent the radioactive detection efficiency of an instrument, with Quantitel **radioactive·detection·efficienciel** (rdctv·dtct·effℓ), equivalent to 1 detectionel per radioactivel (dtctℓ·rdctvℓ$^{-1}$):

(30a)
$$\begin{aligned} \mathcal{N} &= 200_\mathrm{d}\ \mathrm{rdctv}\ell \\ \mathcal{D} &= 20_\mathrm{d}\ \mathrm{dtct}\ell \\ \eta &= \frac{\mathcal{N}}{\mathcal{D}} = 0.1_\mathrm{d}\ \mathrm{rdctv{\cdot}dtct{\cdot}eff}\ell \end{aligned}$$

Notice however, that these Quantitels aren't simply "generic"; they're actually metrology-independent. A count is a count, in any metrology. So these units would not need brand prefixes. Moreover, a ratio of two counts (even two counts of different dimensions), is also going to be metrology-independent.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> Conversion between the count rate and the decay rate may be made using the detection efficiency as a conversion factor. For this detector, if a count rate of $\mathcal{Q}$ = 73 cnt/s is observed, it indicates a decay rate Γ given by
>
> (31) $$\Gamma = \frac{\mathcal{Q}}{\eta} = \frac{73\ \text{cnt}}{0.1\ \text{cnt/dcy}} = 730\frac{\text{dcy}}{\text{s}}$$

So now they've tossed in the dimension of time, to introduce *detection rate* as a dimension. However, I would describe the units for that in terms of (generic) Quantitels, as **detection·ratels** (dtct·rtℓ), which would be equivalent to detectionels per timel. Likewise, they're introducing radioactive decay rate as a dimension, which I'd measure in terms of (generic) **radioactive·ratels** (rdctv·rtℓ), which would be equivalent to radiactivels per timel. Since this depends on the specific timel of a metrology, I'd add brand prefixes and brand mark icons as qualifiers.

So for SI, we'd have **international·detection·ratels** (🌐dtct·rtℓ) and **international·radiactive·ratels** (🌐rdctv·rdℓ) based on the **international·timel** (🌐tmℓ), which of course is the second:

(31🌐)
$$\begin{aligned} \mathcal{Q} &= 73_\mathrm{d}\ \text{🌐dtct·rt}\ell \\ \Gamma &= \frac{\mathcal{Q}}{\eta} = \frac{73_\mathrm{d}\ \text{🌐dtct·rt}\ell}{0.1_\mathrm{d}\ \mathrm{rdctv{\cdot}dtct{\cdot}eff}\ell} = 730_\mathrm{d}\ \text{🌐rdctv·rt}\ell \end{aligned}$$

But these same rates could just as easily be measured in another metrology, such as Primel, in which case we'd be using prime·detection·ratels (⊡dtct·rtℓ) and primel·radioactive·ratels (⊡rdctv·rtℓ) based on the prime·timel (⊡tmℓ):

$$
\begin{aligned}
&\mathcal{Q} = 2.142_{\text{z}}\ \boxdot\text{dtct·rt}\ell \\
&\Gamma = \frac{\mathcal{Q}}{\eta} = \frac{2.142_{\text{z}}\ \boxdot\text{dtct·rt}\ell}{0.12\overline{2497}_{\text{z}}\ \text{rdctv·dtct·eff}\ell} = 19.158_{\text{z}}\ \boxdot\text{rdctv·rt}\ell
\end{aligned} \tag{31$\boxdot$}
$$

Wait what? In this example, isn't the measure coefficient on Γ just supposed to come out to ten times the measure coefficient of $\mathcal{Q}$? Well, look again: It *is* ten times larger, but Primel uses dozenal, where "ten" is spelled "ˊ (a digit), rather than decimal, where "ten" is spelled "10". So multiplying by ten would no longer be a matter of simply shifting the radix point. Once again, we have an example contrived to advantage decimal and SI units. In dozenal, the numbers for $\mathcal{Q}$ and η would look different, despite being exactly equivalent:

$$
\begin{aligned}
&\mathcal{N} = 200_{\text{d}}\ \text{rdct}\ell = 148_{\text{z}}\ \text{rdctv}\ell \\
&\mathcal{D} = 20_{\text{d}}\ \text{dtct}\ell = 18_{\text{z}}\ \text{dtct}\ell \\
&\eta = \frac{\mathcal{N}}{\mathcal{D}} = 0.1_{\text{d}}\ \text{rdctv·dtct·eff}\ell = 0.1\overline{2497}_{\text{z}}\ \text{rdctv·dtct·eff}\ell
\end{aligned} \tag{30b}
$$

If we had contrived the example to advantage dozenal and Primel, we might have had:

$$
\begin{aligned}
&\mathcal{N} = 200_{\text{z}}\ \text{rdct}\ell = 288_{\text{d}}\ \text{rdctv}\ell \\
&\mathcal{D} = 20_{\text{z}}\ \text{dtct}\ell = 24_{\text{d}}\ \text{dtct}\ell \\
&\eta = \frac{\mathcal{N}}{\mathcal{D}} = 0.1_{\text{z}}\ \text{rdctv·dtct·eff}\ell = 0.08\overline{3}_{\text{d}}\ \text{rdctv·dtct·eff}\ell \\
&\mathcal{Q} = 2\ \boxdot\text{dtct·rt}\ell = 69.12_{\text{d}}\ \text{🌐dtct·rt}\ell
\end{aligned} \tag{30c}
$$

So now the dozenal case in Primel looks simply an tidy:

$$
\Gamma = \frac{\mathcal{Q}}{\eta} = \frac{2\ \boxdot\text{dtct·rt}\ell}{0.1_{\text{z}}\ \text{rdctv·dtct·eff}\ell} = 20_{\text{z}}\ \boxdot\text{rdctv·rt}\ell \tag{31a$\boxdot$}
$$

whereas the decimal case in SI now looks messy and complicated:

$$
\Gamma = \frac{\mathcal{Q}}{\eta} = \frac{69.12_{\text{d}}\ \text{🌐dtct·rt}\ell}{0.08\overline{3}_{\text{d}}\ \text{rdctv·dtct·eff}\ell} = 829.44_{\text{d}}\ \text{🌐rdctv·rt}\ell \tag{31a🌐}
$$

Let's take this one step further, and ask, what is the average time *between* radioactive decay events? We could take the reciprocal of the corresponding rate to get a "period":

$$
T_{\Gamma} = \frac{1}{\Gamma} = \frac{1}{20_{\text{z}}\ \boxdot\text{rdctv·rt}\ell} = 0.06_{\text{z}}\ \boxdot\text{tm}\ell/\text{rdctv}\ell = 6\ \boxdot\text{b}{\downarrow}\text{rdctv}\ell\backslash\text{tm}\ell \tag{31b$\boxdot$}
$$

Here I've taken the opportunity to use the SNN power prefix **bicia** to normalize the coefficient and scale the unit down by two dozenal orders of magnitude. Also note the use of the backslash to indicate the reciprocal glue syllable **-ic**, so instead of only being able to express the unit as **primel·bicia·timels per radioactivel**, we can also express it as **primel·bicia·radioactivelic·timels**.

$$
T_{\Gamma} = \frac{1}{\Gamma} = \frac{1}{829.44_{\text{d}}\ \text{🌐rdctv·rt}\ell} \approx 0.0012056327_{\text{d}}\ \text{🌐tm}\ell/\text{rdctv}\ell = 1.2056327_{\text{d}}\ \text{ms/rdctv}\ell \tag{31b🌐}
$$

In this case, we were lucky that we had an SI prefix (**milli**) that could scale the unit down by three powers of ten (to **milliseconds**) so we could normalize the coefficient. If we needed 4 or 5 powers we would have had to go to **microseconds**. Note that, based on my latest formulation of SNN, which includes base-specifying prefixes, even decimal-oriented SI could avail itself of SNN power prefixes: While the default meaning of **tricia** (t↓) is implicitly **zenim·tricia** (z:t↓), i.e. the *zenimal* negative third power, i.e. the negative third power of dozen; on the other hand **decim·tricia** (d:t↓) would mean the decimal negative third power, i.e. the negative third power of *ten*.

So we could refer to this unit as the textbfinternational·decim·tricia·radioactivelic·timel (🌐d:t↓rdctvℓ\tmℓ). And then, if we grant that every metrology has a well-known preferred numeric base, then the base-specifying prefix could be deemed optional, as long as the unit is already qualified with a metrology brand prefix. So in this case, **international·tricia·radioactivelic·timel**

(⊕t↓rdctvℓ\tmℓ) would suffice. And if we needed to go down 4 or 5 powers we could have used **quadcia** or **pentcia** just as easily as **tricia**.

But you have to take some care in interpreting these constants: These express the average time per radioactive decay, but decays are still random events. There is no "cycle" that radioactivity goes through to "generate" each decay. It's not waiting for that period, emiting a particle, then waiting for that period again, emiting another particle, and so forth. Particles are emitted at random, but with a certain statistical concentration in time, i.e., with a frequency—which means "how frequent". There is nothing in the associated "period" that in any way resembles a "phase" that would make this amenable to measuring in angle units!

Mohr and Phillips also consider counts of molecules and other elementary entities:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> Counting also applies to entities such as atoms or molecules. The number density n of molecules in a given volume is the number of molecules $\mathcal{M}$ divided by the volume V
>
> (32) $$n = \frac{\mathcal{M}}{V}$$
>
> which in the current SI has units of m^{-3}. However, this is another case where specification of what the density refers to is useful. This would make the number density consistent with other forms of density, such as mass density or charge density, which have units of $\mathrm{kg/m^3}$ and $\mathrm{C/m^3}$, respectively. For number density, the unit should be $\mathrm{mcl/m^3}$, which follows naturally when $\mathcal{M}$ has the unit mcl, where mcl is the unit for the number of molecules.

But this can be generalized from molecules and atoms to include other particles, including ions and even electrons, and more, lumping all of them together as "elementary entities". Mohr and Phillips use this as a springboard for considering "amount of substance" as a dimension.

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> For macroscopic numbers of molecules or atoms, it is convenient to use the unit mole or mol, where
>
> (33) $$1\ \mathrm{mol} = 6.02 \cdots \times 10^{23}\ \mathrm{ent}$$
>
> where ent is the suggested symbol for entity. This expression makes it clear that the mole, which is the unit of amount of substance, is not just a number, but a number of entities.

The obvious choice for a Quantitel name for the coherent unit of *elementary entity counting* would be the **entitel** (entℓ), representing 1 entity. This would not be just a generic Quantitel, but another metrology-independent one.

The (generic) Quantitel for the *amount of substance* dimension is the **substancel** (sbℓ). Hence, the International System **mole** is the **interational·substancel** (⊕sbℓ):

(31⊕) $$⊕\mathrm{sb}\ell = 6.02214076\times10_{\mathrm{d}}^{23}\ \mathrm{ent}\ell = 602{,}214{,}076\times10_{\mathrm{d}}^{15}\ \mathrm{ent}\ell$$

The 1203$_z$ (2019$_d$) redefinition of SI set the mole to this exact number of entities, but this still approximates the original definition, which was the number of atoms in 12$_d$ **grams** of pure carbon-12$_d$. As you can see, this number is highly biased to be expressed as nine *decimal* digits of significance times a large power of ten. Also, note that the **gram** is not SI's coherent unit of mass, the **kilogram** is.

The **primel·substancel** (⊡sbℓ) is similarly defined as approximately the number of atoms in 10$_z$ **primel·massels** of pure carbon-10$_z$, but we can reasonably set that to an exact number of entitels as well. Needless to say, that number would be nine dozenal digits of significance times a large power of dozen:

(33⊡) ⊡sbℓ = 7.280Ɛ9936×10_{z}^{19} entℓ = 738,0Ɛ9,936×10_{z}^{11} entℓ = 738,0Ɛ9,936$_{z}$ uu↑entℓ

Plus, the **prime·massel** *is* Primel's coherent unit of mass.

While these last two equations are intuitive, there is however a subtle problem with them, because they are a bit misleading. They imply that a **substancel** shares the same dimension as some number of **entitels**, and that therefore the two are commensurate. But the original intent was that "amount of substance" be a distinct dimension, a macroscopic type of quantity, not commensurate with any other dimension, including presumably discrete, microscopic entity counting. It may be better to think of the conversions as being mediated by a constant, which we could call a **substancelic·entitel** (sbℓ\entℓ) constant. Each metrology could have its own version.

The **international·substancelic·entitel** constant, for example, is famously known as "Avogadro's Number", but this gives it some dimensional grounding, rather than treating it as yet another dimensionless number. Mohr and Phillips actually address this later in the paper:

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> The Avogadro constant N_{A} is the number of entities in one mole which can be written as
>
> (54) $N_{\mathrm{A}} = 6.02 \cdots \times 10^{23}$ ent mol^{-1}
>
> in accord with Eq. (33). Evidently, this constant can be viewed as the conversion factor between entities and moles.

This can be recast using Quantitels:

(54🌐) N_{A} = $N_{🌐}$ = 6.02214076×10_{d}^{23} 🌐sbℓ\entℓ = 6.02214076$_{d}$ 🌐bt↑sbℓ\entℓ

Here, the **bt↑** abbreviation stands for the **bitriqua·** prefix, which is the two-digit power prefix for 10_{d}^{23}, with the **decim·** base-specifying prefix assumed because we're using the **International** metrology (🌐).

In Primel, the **primel·substancelic·entitel** constant would be:

(54⊡) $N_{⊡}$ = 7.280Ɛ9936×10_{z}^{19} ⊡sbℓ\entℓ = 7.280Ɛ9936$_{z}$ ⊡ue↑sbℓ\entℓ

In this case, the **ue↑** abbreviation stands for the **unennqua·** prefix, which is the two-digit power prefix for 10_{z}^{19}, with the **zenim·** base-specifying prefix assumed because we're using the **Primel** metrology (⊡).

There is no need to call *N*⊡ "Kodegadulo's number", any more than there was ever a need to call *N*🌐 "Avogadro's number." Does knowing that someone named "Avogadro" is credited with coming up with this number give any clue as to what that number means? Primel is not about aggrandizing *anybody*, not even me. It's about *dimensional clarity*.

Fundamental Constants

Mohr and Phillips apply the concepts previously discussed to take a new look at certain fundamental constants called out by SI:

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> Fundamental constants are parameters in the equations that describe physical phenomena and have the units that are necessary for dimensional consistency. The CODATA recommended values and units for the constants[1]0 are based on the conventions of the current SI, and any modifications of those conventions will have consequences for the units. For example, the equation

> (35) $E = \hbar\omega$
>
> relates E, the energy of a photon, with its angular frequency ω. These quantities are related through the Planck constant $\hbar$, and for the equation to be dimensionally consistent, taking into account the modifications of the SI under consideration, t he unit of $\hbar$ must be J s rad^{-1}, or more suggestively, J/(rad s^{-1}). This is in contrast with the CODATA tabulated value for $\hbar$ which has the unit J s.

First, to "take into account the modifications …under consideration", I would rewrite this equation to annotate the dimensional corrections needed:

(35b) $E = \underset{\smile}{\hbar}\overset{\frown}{\omega}$

And then I would clarify what the actual dimensionalities of $\underset{\smile}{\hbar}$ and $\overset{\frown}{\omega}$ are:

$\underset{\smile}{\hbar}$ = **Planck angular momentum**
Dimension: angular·momentum = angular·mass × angular·velocity = action / true·angle = energy / angular·velocity
Units: ang·momentumel = ang·massel × ang·velocitel = actionel / true·radian = energel / ang·velocitel
(∡mmℓ = ∡msℓ× ∡vcℓ= actℓ / ⓣrad = ngℓ / ∡vcℓ)

$\overset{\frown}{\omega}$ = **angular velocity**
Dimension: angular·velocity = true·angle / time
Units: ang·velocitel = true·radian / timel
(∡vcℓ = ⓣrad / tmℓ)

Note that the dimension of action is equivalent to energy × time, or momentum × length. So:
Units: actionel = energel × timel = momentumel × lengthel.
(actℓ = ngℓ·tmℓ= mmℓ·lgℓ)

Of course, SI lacks named units for any of these. Instead, it has to resort to the usual "derived unit" formulas, which completely obfuscate what it is they measure. But this can be cured using Quantitels:

- 🌐mmℓ= **international·momentumel** = kg·m·s^{-1} = SI coeherent unit of *momentum*
- 🌐actℓ= **international·actionel** = kg·m·^{2}s^{-1} = SI coeherent unit of *action*
- 🌐∡mmℓ= **international·ang·momentumel** = kg·m^{2}·s^{-1}·ⓣrad^{-1} = J·s·ⓣrad = SI coeherent unit of *angular momentum*
- 🌐∡vcℓ= **international·ang·velocitel** = ⓣrad^{2}s^{-1} = SI coeherent unit of *angular velocity*

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> Similarly, the equation
>
> (36) $E = h\nu$
>
> where ν is the photon frequency in hertz, implies that the unit for h is J Hz^{-1}. Both J s rad^{-1} and J Hz^{-1} reduce to Js in the current SI, but they are distinct when units are treated consistently.

But the only way to perform dimensional repair on this equation in the same way as for equation (35) would be to add in **true·turn** units where needed:

(36a) $E = \frac{h}{\text{ⓣtr}}(\nu \cdot \text{ⓣtr}) = \underset{\odot}{h}\overset{\odot}{\nu}$

Here, I've contrived a scheme to annotate dimensional repair by placing a $\odot$ symbol on top of an original symbol to indicate multiplication by a true·turn, or under the original symbol to indicate division by a true·turn. However, note the following:

$$\text{(36b)} \qquad \underset{\odot}{h} = \frac{h}{\text{ⓣtr}} = \frac{h}{\tau\ \text{ⓣrad}} = \frac{\hbar}{\text{ⓣrad}} = \underset{\smile}{\hbar}$$

So $\underset{\odot}{h}$ is just a redundant symbol for $\underset{\smile}{\hbar}$.

$$\text{(36C)} \qquad \overset{\odot}{\nu} = \nu \cdot \text{ⓣtr} = \nu \cdot (\tau\ \text{ⓣrad}) = (\tau\ \nu) \cdot \text{ⓣrad} = \omega \cdot \text{ⓣrad} = \overset{\frown}{\omega}$$

And $\overset{\odot}{\nu}$ is just a redundant symbol for $\overset{\frown}{\omega}$.

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> The two expressions for the photon energy for a given frequency imply
>
> $$\text{(37)} \qquad \hbar\omega = h\nu$$

Even without dimensional correction, this equation is still manifestly true. However, it's much more understandable to say:

$$\text{(37a)} \qquad \underset{\smile}{\hbar}\overset{\frown}{\omega} = \underset{\odot}{h}\overset{\odot}{\nu}$$

which dimensionally corrects both expressions, and is undoubtedly still true — but now is redundant!

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> …and together with Eq. (17) lead to the conventional relation
>
> $$\text{(38)} \qquad \{\hbar\} = \frac{\{h\}_{\mathrm{J \cdot Hz^{-1}}}}{2\pi}$$
>
> between the numerical values of the Planck constant expressed in different units. One often sees
>
> $$\text{(39)} \qquad \hbar \overset{?}{=} \frac{h}{2\pi}$$
>
> but as before, when Eq. (39) is treated as an equality, what is meant is Eq. (38).

But Eq. (39) is correct under the status-quo definitions of h and $\hbar$ without the dimensional corrections I've advocated. If we simply leave those definitions alone, and substitute the annotated symbols I've advocated here, there would be no need to explain all this by resorting to the "pedantic" forms based on the $q = \{q\}[q]$ notation.

> **Mohr & Phillips 11ƐƐ$_{z}$ (2015$_{d}$) wrote:**
>
> Another basic constant involving $\hbar$ is the reduced Compton wavelength of the electron $\lambda\!\!\!^{-}_{\mathrm{C}}$ given by
>
> $$\text{(40)} \qquad \lambda\!\!\!^{-}_{\mathrm{C}} = \frac{\hbar}{m_{\mathrm{e}}c}$$
>
> which has the units
>
> $$\text{(41)} \qquad [\lambda\!\!\!^{-}_{\mathrm{C}}] = \frac{[\hbar]}{[m_{\mathrm{e}}][c]} = \mathrm{m\ rad^{-1}}$$
>
> consistent with Eq. (22).

But here we have another example of a measurement with the dimensionality of *radiality*. Rather than refer to this as a “reduced Compton wavelength”, perhaps this should simply be called the “wave·radiality of the electron”, and express it in dimensionally-corrected form as:

(40a) $\quad \underset{\smile}{ƛ}_{\mathrm{C}} = \dfrac{\underset{\smile}{\hbar}}{m_{\mathrm{e}} c}$

> **Mohr & Phillips 11ƐƐ$_{\mathrm{z}}$ (2015$_{\mathrm{d}}$) wrote:**
>
> Similarly, the Bohr radius a_0 is related to the reduced Compton wavelength by
>
> (42) $\quad ƛ_{\mathrm{C}} = \alpha a_0$
>
> where α is the dimensionless fine-structure constant, so that
>
> (43) $\quad [a_0] = \mathrm{m\ rad^{-1}}$
>
> which is consistent with the use of the angular radius of curvature for mechanical rotational motion.

Again, we have a quantity with the dimension of *radiality*. So perhaps this should be renamed the “Bohr radiality” ($\underset{\smile}{a}_0$) and described in dimensionally-corrected form by:

(42b) $\quad \underset{\smile}{ƛ}_{\mathrm{C}} = \alpha \underset{\smile}{a}_0$

> **Mohr & Phillips 11ƐƐ$_{\mathrm{z}}$ (2015$_{\mathrm{d}}$) wrote:**
>
> For the Rydberg constant, the definition
>
> (44) $\quad R_\infty = \dfrac{\alpha}{4\pi a_0}$
>
> suggests that
>
> (45) $\quad [R_\infty] = \mathrm{cyl\ m^{-1}}$

This suggests to me the dimensional correction $\overset{\odot}{R}_\infty$ to make the units ⓣtr·lgℓ$^{-1}$ (in generic Quantitels) …but true·turns would be a non-coherent angle unit. That explains the extra factor of τ in Eq. (44).

> **Mohr & Phillips 11ƐƐ$_{\mathrm{z}}$ (2015$_{\mathrm{d}}$) wrote:**
>
> …in order to be consistent with the Rydberg formula
>
> (46) $\quad \dfrac{1}{\lambda} = R_\infty \left(\dfrac{1}{{n_1}^2} - \dfrac{1}{{n_2}^2} \right)$
>
> A corresponding angular version of the Rydberg constant is given by
>
> (47) $\quad \text{Ɍ}_\infty = \dfrac{\alpha}{2a_0}$
>
> with units rad m^{-1}, where
>
> (48) $\quad \{\text{Ɍ}_\infty\} = 2\pi\{R_\infty\}$

Which now immediately suggests to me the dimensional correction:

$$(47a) \qquad \overset{\frown}{\bar{R}}_\infty = \frac{\alpha}{2\underset{\smile}{a_0}}$$

eliminating that extra factor of τ, and showing that this constant is actually an example of a *wave-curvature*, which then suggests the following change to the Rydberg formula:

$$(46a) \qquad \frac{1}{\underset{\smile}{\lambda\!\!\!^{-}}} = \overset{\frown}{\bar{R}}_\infty \left(\frac{1}{{n_1}^2} - \frac{1}{{n_2}^2} \right)$$

But note that

$$(48a) \qquad \overset{\frown}{\bar{R}}_\infty = \overset{\odot}{R}_\infty$$

In other words, once both of these have been dimensionally corrected, they become identical, making $\overset{\odot}{R}_\infty$ redundant, and making it unnecessary to resort to a non-coherent angle unit. Then the pedantic nicety of Eq. (48) becomes unnecessary as well.

Mohr and Phillips then go back to the Avogadro constant and give us Eq. (54), which I've already addressed. They continue:

> **Mohr & Phillips 11ƐƐ$_z$ (2015$_d$) wrote:**
>
> It also provides the relation between the molar gas constant $R = 8.31 \cdots \text{ J mol}^{-1}\text{ K}^{-1}$ and the Boltzmann constant k, which is thus given by
>
> $$(55) \qquad k = \frac{R}{N_\text{A}} = 1.38 \cdots \times 10^{-23} \text{ J K}^{-1}\text{ ent}^{-1}$$

In generic Quantitels, a quotient of a **heatel** (htℓ), a coherent unit of *heat* (synonym of **energel**) divided by a **temperaturel** (tpℓ), a coherent unit of *temperature*, gives a **heatabilitel** (htbℓ), a coherent unit of *heatability* or *thermal capacity*. This is an extensive property of a given amount of matter.

If we divide that by some quantity that describes the amount of matter we have, then it becomes an intensive property of the substance. If we use mass to measure our amount of matter, and divide our **heatabilitel** by our **massel** (msℓ), a coherent unit of mass, then we get a **masselic·heatabilitel** (msℓ\htbℓ), a coherent unit of *massic heatability* or *specific thermal capacity*. Primel (like TGM before it) chose to use a representative value for the massic heatability of *water* as its **prime·masselic·heatabilitel** (⊡msℓ\htbℓ) in order to derive its **prime·temperaturel** (⊡tpℓ).

If we divide the heatability of a sample of a gas (assuming it's an ideal gas) by the number of molecules (entities) in that sample, we get an *entitic heatability* which turns out to be a universal constant known as the **Boltzmann constant** (k_B). This would be measured in **entitelic·heatabilitels** (entℓ\htbℓ).

$$(55⊕) \qquad \begin{aligned} &\text{⊕entℓ\\htbℓ} = \text{⊕htℓ ⊕tpℓ}^{-1}\text{ entℓ}^{-1} = \text{J K}^{-1}\text{ entℓ}^{-1} \\ &k_\text{B} = 1.380649 \times 10_\text{d}^{-23}\text{ ⊕entℓ\\htbℓ} = 1.380649_\text{d}\text{ ⊕bt↓entℓ\\htbℓ} \end{aligned}$$

So the Boltzmann constant equals "one decimal three eight zero six four nine international·bitricia·entitelic·heatabilitels".

$$(55⊡) \qquad \begin{aligned} &\text{⊡entℓ\\htbℓ} = \text{⊡htℓ ⊡tpℓ}^{-1}\text{ entℓ}^{-1} \\ &k_\text{B} = 3.3590423 \times 10_\text{z}^{-1↊}\text{ ⊡entℓ\\htbℓ} = 3.3590423_\text{z}\text{ ⊡ud↓entℓ\\htbℓ} \end{aligned}$$

So the Boltzman constant also equals "three zenimal three five nine zero four two three prime·undeccia·entitelic·heatabilitels".

With SI's "molar gas constant", the amount of substance in moles is used to measure the matter in question, in this case any gas. In generic Quantitels we'd divide our **heatabilitel** by our **substancel** (sbℓ), a coherent unit of amount of substance, to get a

substancelic·heatabilitel (sbℓ\tbℓ), a coherent unit of *substancic heatability* or *substancic thermal capacity*. But every metrology can have their own **substancelic·heatabilitel** unit, because each can have their own **substancel** unit. So while the Boltzmann constant is a universal constant, R is a metrology-specific constant, and should be qualified by metrology:

$$R_{\text{🌐}} = N_{\text{🌐}} \cdot k_{\mathrm{B}} = 8.31446261815324_{\mathrm{d}}\ \text{🌐sbℓ\htbℓ}$$
$$R_{\boxdot} = N_{\boxdot} \cdot k_{\mathrm{B}} = 1.\text{Ɛ}91995001\text{Ɛ}581_{\mathrm{z}}\ \boxdot\text{sbℓ\htbℓ} \tag{55b}$$

This discussion continues in Article Review: Mohr, et al., 1206_{z} (2022_{d}).