Article Review: Mohr, et al., 1206_z (2022_d)

by John Volan

This document was adapted from the Article Reviews about Angular Units thread at the DozensOnline forum on tapatalk, in which I review journal articles supporting my assertion that angle should be treated as a first-class physical dimension, and that the coherent unit of angle can only be the radian, not the turn.

On the Dimensions of Angles and Their Units, by Peter J. Mohr, Eric Shirley, and William D. Phillips, *National Institute of Standards and Technology, Gaithersburg, MD 20899, USA*; Michael Trott, *Wolfram Inc., 100 Trade Center Drive, Champaign, IL 61820-7237*, ($1206\text{-}04\text{-}05_z$ | $2022\text{-}04\text{-}05_d$),[1] https://arxiv.org/pdf/2203.12392v2.

Quoting from the abstract:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> We examine implications of angles having their own dimension, in the same sense as do lengths, masses, etc. The conventional practice in scientific applications involving trigonometric or exponential functions of angles is to assume that the argument is the numerical part of the angle when expressed in units of radians. It is also assumed that the functions are the corresponding radian-based versions. These (usually unstated) assumptions generally allow one to treat angles as if they had no dimension and no units, an approach that sometimes leads to serious difficulties.

As I've argued in the Reduced Planck Constant thread on the DozensOnline forum on tapatalk, as well as my previous Article Review: Mohr & Phillips $11ƐƐ_z$ (2015_d), I do not want to touch this conventional practice of treating "angles" as essentially pure numbers and therefore dimensionless, at least within the domain of pure mathematics. I'd rather introduce the concept of a "true·angle" dimension, with units of true·radian, true·turn, etc, and strongly mark when that is being used, within the domain of physics and other physical sciences. I want the "complete" versions of the exponential and trigonometric functions to be strongly-marked as working with true·angles rather than dimensionless angles, and then internally implement those as using the standard dimensionless versions of the functions, after stripping their arguments of dimension.

> **Mohr et al 1206_z (2022_d) wrote:**
>
> Here we consider arbitrary units for angles and the corresponding generalizations of the trigonometric and exponential functions. Such generalizations make the functions complete, that is, independent of any particular choice of unit for angles. They also provide a consistent framework for including angle units in computer algebra programs.

Although the authors here have moved in the right direction, I think given the status quo, they still risk perpetuating the same confusion about angular measures that exists today, unless they adopt something similar to my strategy to disambiguate the notation.

Unit Notation

The authors begin with the tried-and-true notation for describing a dimensioned quantity Q as the product of a dimensionless coefficient $\{Q\}$ and a unit $[Q]$ that bears the dimensionality of the quantity.

[1] This document annotates dozenal numbers with a subscript "z", and decimal numbers with a subscript "d". See https://dozenal.org/article-volan-base-annotation-schemes.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (1) $Q = \{Q\}[Q]$

But they quickly clarify that value of the coefficient is dependent on which unit $[Q]$ has been chosen for this breakdown, so they qualify the coefficient as $\{Q\}_{[Q]}$:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (2) $Q = \{Q\}_{[Q]}[Q]$

And they stress that in principle a dimension could include more than one unit that bears that dimension, so the same quantity Q could be broken down in multiple ways, all of which are still equal and still representative of the same quantity:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (3) $Q = \{Q\}_{[Q]_1}[Q]_1 = \{Q\}_{[Q]_2}[Q]_2$

And as an example, they consider a simple case of a length that could be described as 3 meters or as (approximately) 118.11$_d$ inches.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (4) $L = 3\ \mathrm{m} = 118.11\cdots\ \mathrm{in}$
>
> where $[L]_1 = \mathrm{m}$, $\{L\}_{\mathrm{m}} = 3$, $[L]_2 = \mathrm{in}$, $\{L\}_{\mathrm{in}} = 118.11\cdots$

Units for Angles

The authors then go on to apply this idea to angles:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (5) $\theta = \{\theta\}_{[\theta]}\,[\theta]$

And then they consider an example of an octant or diagonal angle, which can be measured either as 45$_d$ degrees, or as $\frac{\tau}{8}$ radians, or as $\frac{1}{8}$ turn:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (6) $\theta = 45\ \mathrm{deg}$ where $[\theta] = \mathrm{deg}$ and $\{\theta\} = 45$
>
> (7) $\theta = \frac{\pi}{4}\ \mathrm{rad}$ where $[\theta] = \mathrm{rad}$ and $\{\theta\} = \frac{\pi}{4}$
>
> (8) $\theta = \frac{1}{8}\ \mathrm{rev}$ where $[\theta] = \mathrm{rev}$ and $\{\theta\} = \frac{1}{8}$

(It seems that even after another 7 years, Mohr and his colleagues at NIST still haven't got the memo about $\tau = 2\pi$.)

It's all very well that the authors here can describe how to break down an angle measurement using three different units, but I think they are already setting themselves up for trouble here. Because the mathematical world, as well as anyone using the International System of Units (SI), is still going to be stuck thinking of angles as dimensionless numbers, with the radian unit itself being literally equal to the dimensionless pure number 1:

(K1) $\text{rad} = 1$

If they take this seriously, then *all* angle units must similarly be literally equal to some pure dimensionless real number. So a turn unit is literally identical to the pure dimensionless transcendental circle constant τ:

(K2) $\text{tr} = \tau\ \text{rad} = \tau \cdot 1 = \tau \approx 6.28318531_\text{d} \approx 6.34941697_\text{z}$

And likewise the degree unit is *literally identical* to the number τ divided by 360_d:

(K3) $\text{deg} = \dfrac{\text{tr}}{360_\text{d}} = \dfrac{\tau}{360_\text{d}} \approx 0.01745329_\text{d} \approx 0.0261\text{↊↋}0_\text{z}$

The degree unit represents the ancient and mainstream-mathematics attempt to divide the turn into rational subunits. Unfortunately, it resorts to using sexagesimal base to get high divisibility. That isn't in keeping the the "metric" principle that the prevailing numeric base should be used for subdividing units. But a purely decimal division of the circle would be a disaster; even the division by four hundred of the gradian never got much traction. On the other hand, if dozenal were the prevailing base, then we'd get ample divisibility while staying within the "metric" principle. As I've described elsewhere (including this post from the Reduced Planck Constant thread), a dozenal approach to angles could take advantage of the scaling power prefixes from SNN to divide the turn into **uncia·turns**, **bicia·turns**, **tricia·turns**, etc.

If we take the **tricia·turn** (t↓tr) as an example dozenal unit that might replace our use of degrees, then we can express our example octant as:

(6⊡) $\theta = 160_\text{z}\ \text{t↓tr} \qquad \text{where} \qquad [\theta] = \text{t↓tr} \qquad \text{and} \qquad \{\theta\} = 160_\text{z}$

The tricia·turn is just another dimensionless unit. So by the above argument, it would be *literally identical* to a pure number as well, in this case the number $0.001_\text{z}\ \tau$:

(K3⊡) $\text{t↓tr} = \text{t↓}\tau = 0.001_\text{z}\ \tau \approx 0.00634941697_\text{z}$

The authors argue later that selection of units for a dimension is generally an arbitrary choice. Despite that, radians as the unit for dimensionless angles is not arbitrary, but demanded by the way trigonometry and calculus and other branches of mathematics actually work. It underlies the handling of the unit circle on the complex number plane and therefore impacts the very definition of *number*! From a mathematical perspective, *angles* are *simply numbers*.

As a consequence of this, if we start from Eq. (7) and substitute the value for rad from Eq. (K1), we get our octant angle θ literally equivalent to the pure number $\frac{\tau}{8}$:

(7a) $\theta = \dfrac{\tau}{8}\text{rad} = \dfrac{\tau}{8} \cdot 1 = \dfrac{\tau}{8} \approx 0.78539816_\text{d} \approx 0.95120242_\text{z}$

If we start from Eq. (8) and substitute the value for rev = tr from Eq. (K2), we again get the same θ literally equivalent to the pure number $\frac{\tau}{8}$:

(8a) $\theta = \dfrac{1}{8}\text{tr} = \dfrac{1}{8} \cdot \tau = \dfrac{\tau}{8} \approx 0.78539816_\text{d} \approx 0.95120242_\text{z}$

And likewise, if we start from Eq. (6) and substitute the value for deg from Eq. (K3), we get the same θ once more literally equivalent to the pure number $\frac{\tau}{8}$:

(6a) $\theta = 45_\text{d}\ \text{deg} = 45_\text{d} \dfrac{\tau}{360_\text{d}} = \dfrac{\tau}{8} \approx 0.78539816_\text{d} \approx 0.95120242_\text{z}$

For the dozenalists, if we start from Eq. (6⊡) and substitute the value for t↓tr from Eq. (K3⊡), we get the same θ yet again, literally equivalent to the pure number $\frac{\tau}{8}$:

(6⊡a) $\theta = 160_z\ \text{t}{\downarrow}\text{tr} = 160_z \cdot 0.001_z\ \tau = 0.16_z\ \tau = \frac{\tau}{8} \approx 0.95120242_z \approx 0.78539816_d$

No matter what we do, in every case we wind up with a pure, dimensionless number that indicates the number of radians in the angle. In terms of the notation from Eq. (1), θ is already the "the numerical part of the angle when expressed in units of radians". But if we truly wish to express the angle as a *dimensioned* quantity, we need to assert **true·angle** as a dimension, with units such as **true·radian** (ⓣrad), **true·turn** (ⓣtr), and **true·degree** (ⓣdeg), that bear that dimension. And then we need some annotation or operator to mark when we are introducing a factor of true·angle into a variable or expression, either by multiplication by a **true·radian**:

(K4) $\overset{\frown}{x} = x \cdot \text{ⓣrad}$

or by division by a **true·radian**:

(K54) $\underset{\smile}{x} = \frac{x}{\text{ⓣrad}}$

But notice something interesting. If we truly take seriously what Eq. (K1) is saying, then the true·radian operator in Eq. (K4) now offers new ways to symbolize the true·radian itself!

(K1a) $\text{rad} = 1 \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\text{rad}} = \overset{\frown}{1} = \text{ⓣrad} \quad (!)$

And if we take serious what Eq. (K2) is saying, we even get new ways to symbolize the true·turn!

(K2a) $\text{tr} = \tau \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\text{tr}} = \overset{\frown}{\tau} = \text{ⓣtr} \quad (!)$

We even get new ways to symbolize the true·degree, if we take seriously what Eq. (K3) is saying!

(K3a) $\text{deg} = \frac{\tau}{360_d} \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\text{deg}} = \frac{\overset{\frown}{\tau}}{360_d} = \text{ⓣdeg} \quad (!)$

And for the dozenalists here, we get new ways to symbolize the tricia·true·turn, if we take seriously what Eq. (K3⊡) is saying:

(K3⊡a) $\text{t}{\downarrow}\text{tr} = \text{t}{\downarrow}\tau \quad \overset{\frown}{\Longrightarrow} \quad \text{t}{\downarrow}\overset{\frown}{\text{tr}} = \text{t}{\downarrow}\overset{\frown}{\tau} = \text{t}{\downarrow}\text{ⓣtr} \quad (!)$

So, going back to Eq. (7a), we can derive a true·angle variable $\overset{\frown}{\theta}$ by using the true·radian operator to effectively multiply dimensionless-number θ by 1 true·radian:

(7b) $\theta = \frac{\tau}{8}\text{rad} = \frac{\tau}{8} \cdot 1 = \frac{\tau}{8} \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\theta} = \frac{\tau}{8}\overset{\frown}{\text{rad}} = \frac{\tau}{8} \cdot \overset{\frown}{1} = \frac{\overset{\frown}{\tau}}{8}$

We can similarly derive the same true·angle $\overset{\frown}{\theta}$ by going back to Eq. (8a) and applying the true·radian operator there:

(8b) $\theta = \frac{1}{8}\text{tr} = \frac{\tau}{8} \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\theta} = \frac{1}{8}\overset{\frown}{\tau} = \frac{\overset{\frown}{\tau}}{8}$

We can even derive true·angle $\overset{\frown}{\theta}$ by going back to Eq. (6a) and applying the true·radian operator there:

(6b) $\theta = 45_d\ \text{deg} \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\theta} = 45_d\ \overset{\frown}{\text{deg}} = 45_d \cdot \frac{\overset{\frown}{\tau}}{360_d} = \frac{\overset{\frown}{\tau}}{8}$

For the dozenalists here, we can derive true·angle $\overset{\frown}{\theta}$ yet again by going back to Eq. (6⊡a) and applying the true·radian operator there:

(6⊡b) $\theta = 160_z\ \text{t}{\downarrow}\text{tr} \quad \overset{\frown}{\Longrightarrow} \quad \overset{\frown}{\theta} = 160_z\ \text{t}{\downarrow}\overset{\frown}{\text{tr}} = 160_z \cdot 0.001_z\ \overset{\frown}{\tau} = \frac{\overset{\frown}{\tau}}{8}$

It is only at this point that Eqs. (1) and (2) truly become relevant, because now $\overset{\frown}{\theta}$ is a variable with true dimensionality. From Eq. (7a) we get this unit decomposition for $\overset{\frown}{\theta}$:

(2a) $\{\overset{\frown}{\theta}\}_{\overset{\frown}{\mathrm{rad}}} = \frac{\tau}{8} \qquad [\overset{\frown}{\theta}] = \overset{\frown}{\mathrm{rad}} = \overset{\frown}{1} \qquad \overset{\frown}{\theta} = \theta\,\overset{\frown}{\mathrm{rad}} = \frac{\tau}{8}\,\overset{\frown}{\mathrm{rad}}$

From Eq. (8a) we get a different unit decomposition for $\overset{\frown}{\theta}$

(2b) $\{\overset{\frown}{\theta}\}_{\overset{\frown}{\mathrm{tr}}} = \frac{1}{8} \qquad [\overset{\frown}{\theta}] = \overset{\frown}{\mathrm{tr}} = \overset{\frown}{\tau} \qquad \overset{\frown}{\theta} = \{\overset{\frown}{\theta}\}_{\overset{\frown}{\mathrm{tr}}}\,[\overset{\frown}{\theta}] = \frac{1}{8}\,\overset{\frown}{\mathrm{tr}}$

And from Eq. (6a) we get yet another unit decomposition:

(2c) $\{\overset{\frown}{\theta}\}_{\overset{\frown}{\mathrm{deg}}} = 45_{\mathrm{d}} \qquad [\overset{\frown}{\theta}] = \overset{\frown}{\mathrm{deg}} = \frac{\overset{\frown}{\tau}}{360_{\mathrm{d}}} \qquad \overset{\frown}{\theta} = \{\overset{\frown}{\theta}\}_{\overset{\frown}{\mathrm{deg}}}\,[\overset{\frown}{\theta}] = 45_{\mathrm{d}}\,\overset{\frown}{\mathrm{deg}}$

From Eq. (6⊡b) we get a unit decomposition for dozenalists to love:

(2d) $\{\overset{\frown}{\theta}\}_{\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}} = 160_{\mathrm{z}} \qquad [\overset{\frown}{\theta}] = \mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}} = 0.001_{\mathrm{z}}\,\overset{\frown}{\tau} \qquad \overset{\frown}{\theta} = \{\overset{\frown}{\theta}\}_{\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}}\,[\overset{\frown}{\theta}] = 160_{\mathrm{z}}\,\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}$

But notice something interesting about Eqs. (7b) and (8b): They both say

(7b|8b) $\{\overset{\frown}{\theta}\} = \frac{\overset{\frown}{\tau}}{8}$

but Eq. (7b) interprets that as “true·angle theta equals tau eighths true·radians”, whereas Eq. (8b) interprets that as “true·angle theta equals one eighth true·turn”.

Combining Eqs. (6b-8b), we can assert that each of these expresses the same true·angle value, but each using a different $\{\overset{\frown}{\theta}\}$ and $[\overset{\frown}{\theta}]$, thus demonstrating Eq. (3):

(K6) $\{\overset{\frown}{\theta}\} = \frac{\tau}{8}\overset{\frown}{\mathrm{rad}} = \frac{1}{8}\overset{\frown}{\mathrm{tr}} = 45_{\mathrm{d}}\,\overset{\frown}{\mathrm{deg}} = 160_{\mathrm{z}}\,\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}$

To convert between a true·radian and any other true·angle unit, we can keep in mind unit equalities such as:

(K7) $1\,\overset{\frown}{\mathrm{tr}} = \tau\,\overset{\frown}{\mathrm{rad}} = 360_{\mathrm{d}}\,\overset{\frown}{\mathrm{deg}} = 1000_{\mathrm{z}}\,\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}$

From those, we can derive unit conversion factors that are all equal to 1 (the multiplicative identity) that we can apply as needed:

(K8) $1 = \frac{1\,\overset{\frown}{\mathrm{tr}}}{\tau\,\overset{\frown}{\mathrm{rad}}} = \frac{\tau\,\overset{\frown}{\mathrm{rad}}}{1\,\overset{\frown}{\mathrm{tr}}} = \frac{\tau\,\overset{\frown}{\mathrm{rad}}}{360_d\,\overset{\frown}{\mathrm{deg}}} = \frac{360_d\,\overset{\frown}{\mathrm{deg}}}{\tau\,\overset{\frown}{\mathrm{rad}}} = \frac{\tau\,\overset{\frown}{\mathrm{rad}}}{1000_z\,\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}} = \frac{1000_z\,\mathrm{t\downarrow}\overset{\frown}{\mathrm{tr}}}{\tau\,\overset{\frown}{\mathrm{rad}}}$

Next, the authors consider trigonometric functions, and imagine multiple versions of these functions, each taking a dimensionless argument θ and returning a dimensionless result, but each making a different assumption about what angle unit θ is measured in:

> **Mohr et al 1206_{z} (2022_{d}) wrote:**
>
> (9) $\sin_{\mathrm{deg}}(45) = \frac{1}{\sqrt{2}}$

$$(10)\qquad \sin_{\text{rad}}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$(11)\qquad \sin_{\text{rev}}\left(\frac{1}{8}\right) = \frac{1}{\sqrt{2}}$$

This begs the question of how these different functions would be implemented. Eq. (10) would represent the most simple and direct implementation given the nature of trigonometry. But ultimately, there should be no need to proliferate multiple solutions. If we can instead have a single "complete" function that is able to take a true·angle argument $\overset{\frown}{\theta}$, then no matter which true·angle unit that argument is expressed in, this one function can suffice to get the desired result. It would do this by stripping its argument of units by dividing by true·radian to yield dimensionless θ and then pass that on to the pure mathematical version of the function. This can be defined for each of the relevant trigonometric functions:

$$(\text{K9})\qquad \underset{\smile}{\sin}\,\overset{\frown}{\theta} = \sin\underset{\smile}{\overset{\frown}{\theta}} = \sin\theta$$

$$(\text{K↊})\qquad \underset{\smile}{\cos}\,\overset{\frown}{\theta} = \cos\underset{\smile}{\overset{\frown}{\theta}} = \cos\theta$$

$$(\text{K↋})\qquad \underset{\smile}{\tan}\,\overset{\frown}{\theta} = \tan\underset{\smile}{\overset{\frown}{\theta}} = \tan\theta$$

$$(\text{K10})\qquad \underset{\smile}{\cot}\,\overset{\frown}{\theta} = \cot\underset{\smile}{\overset{\frown}{\theta}} = \cot\theta$$

$$(\text{K11})\qquad \underset{\smile}{\sec}\,\overset{\frown}{\theta} = \sec\underset{\smile}{\overset{\frown}{\theta}} = \sec\theta$$

$$(\text{K12})\qquad \underset{\smile}{\csc}\,\overset{\frown}{\theta} = \csc\underset{\smile}{\overset{\frown}{\theta}} = \csc\theta$$

For example, if we use the "complete" version of sin from Eq. (K9) and pass it $\overset{\frown}{\theta}$ expressed in true·radians as in Eq. (7b), then the implementation is straightforward:

$$(10\text{a})\qquad \underset{\smile}{\sin}\left(\frac{\tau}{8}\,\overset{\frown}{\text{rad}}\right) = \sin\underset{\smile}{\underbrace{\left(\frac{\tau}{8}\,\overset{\frown}{\text{rad}}\right)}} = \sin\left(\frac{\tau}{8}\,\frac{\overset{\frown}{\text{rad}}}{\overset{\frown}{\text{rad}}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

But looking at Eq. (7b) again there's an even more concise way to express this:

$$(10\text{b})\qquad \underset{\smile}{\sin}\left(\frac{\tau}{8}\cdot\overset{\frown}{1}\right) = \sin\left(\frac{\tau}{8}\cdot\underset{\smile}{\overset{\frown}{1}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

If instead we pass in $\overset{\frown}{\theta}$ expressed in true·turns as in Eq. (8b), then we need to enlist one of the unit conversions from Eq. (K8):

$$(11\text{a})\qquad \underset{\smile}{\sin}\left(\frac{1}{8}\,\overset{\frown}{\text{tr}}\right) = \sin\underset{\smile}{\underbrace{\left(\frac{1}{8}\,\overset{\frown}{\text{tr}}\right)}} = \sin\left(\frac{1}{8}\,\frac{\overset{\frown}{\text{tr}}}{\overset{\frown}{\text{rad}}}\right) = \sin\left(\frac{1}{8}\,\frac{\overset{\frown}{\text{tr}}}{\overset{\frown}{\text{rad}}}\cdot\frac{\tau\,\overset{\frown}{\text{rad}}}{\overset{\frown}{\text{tr}}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

But looking at Eq. (8b) again, there's an even more concise way to express Eq. (11a):

$$(11\text{b})\qquad \underset{\smile}{\sin}\left(\frac{1}{8}\,\overset{\frown}{\tau}\right) = \sin\left(\frac{1}{8}\cdot\underset{\smile}{\overset{\frown}{\tau}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

If we pass in $\overset{\frown}{\theta}$ expressed in true·degrees as in Eq. (6b), then we need to enlist another unit conversion from Eq. (K8):

$$(9\text{a})\qquad \underset{\smile}{\sin}\left(45_{\text{d}}\,\overset{\frown}{\text{deg}}\right) = \sin\underset{\smile}{\underbrace{\left(45_{\text{d}}\,\overset{\frown}{\text{deg}}\right)}} = \sin\left(45_{\text{d}}\frac{\overset{\frown}{\text{deg}}}{\overset{\frown}{\text{rad}}}\right) = \sin\left(45_{\text{d}}\frac{\overset{\frown}{\text{deg}}}{\overset{\frown}{\text{rad}}}\cdot\frac{\tau\,\overset{\frown}{\text{rad}}}{360_{\text{d}}\,\overset{\frown}{\text{deg}}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

But looking at Eq. (6b) again, there's an even more concise way to express Eq. (9a):

(9b) $$\underset{\smile}{\sin}\left(45_{\text{d}}\ \overset{\frown}{\text{deg}}\right) = \underset{\smile}{\sin}\left(45_{\text{d}} \cdot \frac{\overset{\frown}{\tau}}{360_{\text{d}}}\right) = \sin\left(\frac{45_{\text{d}}}{360_{\text{d}}} \cdot \underset{\smile}{\overset{\frown}{\tau}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

If we pass in $\overset{\frown}{\theta}$ expressed in true·tricia·turns as in Eq. (6⊡b), then we need to enlist another unit conversion from Eq. (K8):

(9⊡a) $$\underset{\smile}{\sin}\left(160_{\text{z}}\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}\right) = \sin\underset{\smile}{\underbrace{\left(160_{\text{z}}\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}\right)}} = \sin\left(160_{\text{z}} \frac{\text{t}{\downarrow}\overset{\frown}{\text{tr}}}{\overset{\frown}{\text{rad}}}\right) = \sin\left(160_{\text{z}} \frac{\text{t}{\downarrow}\overset{\frown}{\text{tr}}}{\overset{\frown}{\text{rad}}} \cdot \frac{\tau\ \overset{\frown}{\text{rad}}}{1000_{\text{z}}\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

But looking at Eq. (6⊡b) again, there's an even more concise way to express Eq. (9⊡a):

(9⊡b) $$\underset{\smile}{\sin}\left(160_{\text{z}}\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}\right) = \underset{\smile}{\sin}\left(160_{\text{z}} \cdot 0.001_{\text{z}}\ \overset{\frown}{\tau}\right) = \sin\left(0.16_{\text{z}} \cdot \underset{\smile}{\overset{\frown}{\tau}}\right) = \sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

In the end, each of these cases winds up boiling down to the same invocation of the same "pure math" radian-based version of sin:

(K13) $$\sin\left(\frac{\tau}{8}\right) = \frac{1}{\sqrt{2}}$$

For the inverse trigonometric functions, the authors again imagine different versions of each function that take a dimensionless argument and generate the dimensionless coefficient of the angle, assuming different angle units.

> **Mohr et al 1206_{z} (2022_{d}) wrote:**
>
> (12) $$\text{arcsin}_{\text{deg}}\left(\frac{1}{\sqrt{2}}\right) = 45$$
>
> (13) $$\text{arcsin}_{\text{rad}}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$
>
> (14) $$\text{arcsin}_{\text{rev}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{8}$$

They even imagine these functions decorating their results with the angle unit they each presume:

> **Mohr et al 1206_{z} (2022_{d}) wrote:**
>
> (15) $$\text{arcsin}_{\text{deg}}\left(\frac{1}{\sqrt{2}}\right)\ \text{deg} = 45\ \text{deg}$$
>
> (16) $$\text{arcsin}_{\text{rad}}\left(\frac{1}{\sqrt{2}}\right)\ \text{rad} = \frac{\pi}{4}\ \text{rad}$$
>
> (17) $$\text{arcsin}_{\text{rev}}\left(\frac{1}{\sqrt{2}}\right)\ \text{rev} = \frac{1}{8}\ \text{rev}$$

But again, proliferating such function versions for every angle unit is unnecessary. All that is needed is one "complete" version for each inverse trigonometric function:

(K14) $$x = \underset{\smile}{\sin}\,\overset{\frown}{\theta} \qquad \Longleftrightarrow \qquad \overset{\frown}{\text{arcsin}}\,x = \overset{\frown}{\theta}$$

(K15) $x = \underset{\smile}{\cos}\,\overset{\frown}{\theta} \iff \overset{\frown}{\arccos}\,x = \overset{\frown}{\theta}$

(K16) $x = \underset{\smile}{\tan}\,\overset{\frown}{\theta} \iff \overset{\frown}{\arctan}\,x = \overset{\frown}{\theta}$

(K17) $x = \underset{\smile}{\cot}\,\overset{\frown}{\theta} \iff \overset{\frown}{\text{arccot}}\,x = \overset{\frown}{\theta}$

(K18) $x = \underset{\smile}{\sec}\,\overset{\frown}{\theta} \iff \overset{\frown}{\text{arcsec}}\,x = \overset{\frown}{\theta}$

(K19) $x = \underset{\smile}{\csc}\,\overset{\frown}{\theta} \iff \overset{\frown}{\text{arccsc}}\,x = \overset{\frown}{\theta}$

Each of these invokes the "pure math" radian-based implementation of the inverse function, and immediately turns the dimensionless result into a true·angle by simply multiplying it by a true·radian. So Eq. (16) can be immediately accomplished by:

$$\text{(16a)} \quad \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\tau}{8} \quad \Longrightarrow \quad \overset{\frown}{\arcsin}\left(\frac{1}{\sqrt{2}}\right) = \frac{\tau}{8}\overset{\frown}{\text{rad}}$$

Then, to get the result in true·turns as in Eq. (17), we can simply apply one of the conversions from Eq. (K8):

$$\text{(17a)} \quad \overset{\frown}{\arcsin}\left(\frac{1}{\sqrt{2}}\right) = \frac{\tau}{8}\overset{\frown}{\text{rad}} \quad \Longrightarrow \quad \frac{\tau}{8}\overset{\frown}{\text{rad}} \cdot \frac{\overset{\frown}{\text{tr}}}{\tau\ \overset{\frown}{\text{rad}}} = \frac{1}{8}\overset{\frown}{\text{tr}}$$

Likewise, to get the result in true·degrees as in Eq. (15), we can simply apply another one of the conversions from Eq. (K8):

$$\text{(15a)} \quad \overset{\frown}{\arcsin}\left(\frac{1}{\sqrt{2}}\right) = \frac{\tau}{8}\overset{\frown}{\text{rad}} \quad \Longrightarrow \quad \frac{\tau}{8}\overset{\frown}{\text{rad}} \cdot \frac{360_{\text{d}}\ \overset{\frown}{\text{deg}}}{\tau\ \overset{\frown}{\text{rad}}} = 45_{\text{d}}\ \overset{\frown}{\text{deg}}$$

For the dozenalists, yet another one of the conversions from Eq. (K8) can give us the result in tricia·true·turns:

$$\text{(15b)} \quad \overset{\frown}{\arcsin}\left(\frac{1}{\sqrt{2}}\right) = \frac{\tau}{8}\overset{\frown}{\text{rad}} \quad \Longrightarrow \quad \frac{\tau}{8}\overset{\frown}{\text{rad}} \cdot \frac{1000_z\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}}{\tau\ \overset{\frown}{\text{rad}}} = 160_z\ \text{t}{\downarrow}\overset{\frown}{\text{tr}}$$

Unit-Independent Transcendental Functions

At this point, the authors begin assuming that the standard symbols for angles, including unit symbols, actually have true dimension, and start interpreting the standard symbols for transcendental functions as meaning the "complete", unit-independent versions. I think that's unwise because it muddles the discussion. So I will show equivalent equations where those symbols and functions are strongly marked with my operator annotations.

A. Trigonometric Functions

The authors begin by contrasting the situation in physics with that of mathematics, where units of measure and physical dimensions are of no concern, and everything is described in terms of abstract numbers. They highlight how calculus interacts with trigonometry, showing this example of differentiating a sine function to get the cosine function as its rate of change:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (20) “$\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = \cos(\theta)$”

Note the scare quotes here, highlighting the fact that this is not a dimensionally-balanced equation, because the differential on the left has change in angle in the denominator, where the right side does not. But mathematicians dealing only with pure, dimensionless numbers do not care about that. To rectify this, the authors go back to first principles and examine the basic definition of a differential as a limit on a ratio of deltas as the deltas are shrunk to infinitessimals. After several steps, they generate an alternative equation where differentiating pulls out a constant:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (25) $\frac{\mathrm{d}}{\mathrm{d}\theta}\sin(\theta) = \mathcal{C}\ \cos(\theta)$

with the constant determined to be:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (29) $\mathcal{C} = \frac{2\pi}{\Theta}$

where Θ represents “the angle of a complete revolution or period of the sine function”, and the 2π comes from $2\pi r$, “the circumference of a circle of radius r”. Very curiously, the authors proceed to decorate the rest of this paper with this constant, expressed as this ratio, without ever examining what it means or how it can be reduced or simplified.

First of all, they are treating Θ as a true·angle with true dimensionality, so we should annotate that by applying the true·radian operator on it. But then, to keep the equation dimensionally balanced, we’ll need to apply the true·radianic operator on the left-hand side:

(29a) $$\underset{\smile}{\mathcal{C}} = \frac{2\pi}{\overset{\frown}{\Theta}}$$

Next, note that they are still failing to take advantage of using $\tau = 2\pi$. Plus, it’s clear from their definition of Θ, that $\overset{\frown}{\Theta}$ represents a true·turn, which according to Eq. (K2a) can be expressed as $\overset{\frown}{\tau}$. Hence:

(29b) $$\underset{\smile}{\mathcal{C}} = \frac{2\pi}{\overset{\frown}{\Theta}} = \frac{\tau}{\overset{\frown}{\tau}} = \frac{1}{\overset{\frown}{1}} = \underset{\smile}{1} = \overset{\frown}{\mathrm{rad}}^{-1}$$

In other words, $\underset{\smile}{\mathcal{C}}$ is just the true·radianic constant! And of course, its reciprocal — also sprinkled throughout the rest of the article — is just the true·radian constant: which according to Eq. (K1a) can be expressed as $\overset{\frown}{1}$. Hence:

(29c) $$\underset{\smile}{\mathcal{C}}^{-1} = \frac{\overset{\frown}{\Theta}}{2\pi} = \frac{\overset{\frown}{\tau}}{\tau} = \frac{\overset{\frown}{1}}{1} = \overset{\frown}{1} = \overset{\frown}{\mathrm{rad}}$$

To my mind, the authors here have injected unnecessary complexity and obfuscation into something that should be simple and straightforward. No matter how you dress it up in symbolism, this is still just going to be a true·radian.

Going back to Eq. (25), we need to annotate that they are now treating θ as the true·angle $\overset{\frown}{\theta}$, and that where they are using functions sin and cos, they really mean the “complete” versions of those, $\underset{\smile}{\sin}$ and $\underset{\smile}{\cos}$:

(25a) $$\frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\underset{\smile}{\sin}\overset{\frown}{\theta} = \underset{\smile}{\mathcal{C}}\ \underset{\smile}{\cos}\overset{\frown}{\theta}$$

But I had already worked this out a month ago, in this post in the Reduced Planck Constant thread. However, I used a different logical path to get to the solution:

$$\text{(25b)}\qquad \frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\underset{\smile}{\sin}\overset{\frown}{\theta} = \underbrace{\frac{\mathrm{d}\sin\theta}{\mathrm{d}\theta}\cdot\frac{1}{\overset{\frown}{1}}}_{1} = \underbrace{\cos\theta\cdot\underset{\smile}{1}}_{2} = \underbrace{\underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\underset{\smile}{1}}_{3} = \underbrace{\underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-1}}_{4}$$

I've simplified my method for deriving this a bit from what I did in that post, but the steps I used were essentially:

1. Reduce the complete $\underset{\smile}{\sin}$ function and its true·angle argument $\overset{\frown}{\theta}$ to their pure-math equivalents; in the process, pull the true·radian constant out of $\mathrm{d}\overset{\frown}{\theta}$.
2. Differentiate $\sin\theta$ with respect to $\mathrm{d}\theta$ to get $\cos\theta$; and invert the constant.
3. Convert the pure math $\cos\theta$ to its complete version.
4. Convert the constant to a more conventional form, revealing the unit more obviously.

The authors also do this for cosine:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$\text{(32)}\qquad \frac{\mathrm{d}}{\mathrm{d}\theta}\cos(\theta) = -\mathcal{C}\ \sin(\theta)$$

Here's my derivation for that:

$$\text{(32b)}\qquad \frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\underset{\smile}{\cos}\overset{\frown}{\theta} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}\theta}\cdot\frac{1}{\overset{\frown}{1}} = -\sin\theta\cdot\underset{\smile}{1} = -\underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\underset{\smile}{1} = -\underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-1}$$

In fact, I had worked out the first few orders of differentiation of sin, but now I can express them as:

$$\text{(K1ᘔ)}\qquad \frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\underset{\smile}{\sin}\overset{\frown}{\theta} = \underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\underset{\smile}{1} = \underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-1}$$

$$\text{(K1Ɛ)}\qquad \frac{\mathrm{d}^2}{\mathrm{d}\overset{\frown}{\theta}^2}\underset{\smile}{\sin}\overset{\frown}{\theta} = -\underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\underset{\smile}{1}^2 = -\underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-2}$$

$$\text{(K20)}\qquad \frac{\mathrm{d}^3}{\mathrm{d}\overset{\frown}{\theta}^3}\underset{\smile}{\sin}\overset{\frown}{\theta} = -\underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\underset{\smile}{1}^3 = -\underset{\smile}{\cos}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-3}$$

$$\text{(K21)}\qquad \frac{\mathrm{d}^4}{\mathrm{d}\overset{\frown}{\theta}^4}\underset{\smile}{\sin}\overset{\frown}{\theta} = \underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\underset{\smile}{1}^4 = \underset{\smile}{\sin}\overset{\frown}{\theta}\cdot\overset{\frown}{\mathrm{rad}}^{-4}$$

etc …

The powers of true·radian on the right all make sense in terms of the powers of true·angle appearing in the denominators of the differentials on the left.

B. Exponential Function

Next, the authors tackle the transcendental exponential function. However, they recognize that simply raising the base of the natural logarithms e to a power does not constitute a complete function, because this requires the exponent to be a dimensionless number. How to resolve that might not be clear at first. So they approach a definition of the complete function exp by using complete versions of the transcendental trigonometric functions:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (33) $\exp(\mathrm{i}\theta) = \cos(\theta) + \mathrm{i}\sin(\theta)$

But explicitly marking this with annotations makes it more clear that these functions are now complete and defined in terms of true‧angle arguments:

(33a) $$\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,) = \underset{\smile}{\cos}\overset{\frown}{\theta} + \mathrm{i}\,\underset{\smile}{\sin}\overset{\frown}{\theta}$$

Next, the authors look at the derivative of this function with respect to true‧angle:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (34) $\frac{\mathrm{d}}{\mathrm{d}\theta}\exp(\mathrm{i}\theta) = -\mathcal{C}\cos(\theta) + \mathrm{i}\,\mathcal{C}\,\sin(\theta) = \mathrm{i}\,\mathcal{C}\exp(\mathrm{i}\,\theta) = \mathrm{i}\frac{2\pi}{\Theta}\exp(\mathrm{i}\,\theta)$

Again, proper annotation can make this clearer:

(34a) $$\frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,) = \frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\left(\underset{\smile}{\cos}\overset{\frown}{\theta} + \mathrm{i}\,\underset{\smile}{\sin}\overset{\frown}{\theta}\right) = \frac{-\underset{\smile}{\sin}\overset{\frown}{\theta} + \mathrm{i}\,\underset{\smile}{\cos}\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}} = \frac{\mathrm{i}\,\underset{\smile}{\cos}\overset{\frown}{\theta} + \mathrm{i}^2\,\underset{\smile}{\sin}\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}} = \frac{\mathrm{i}\left(\underset{\smile}{\cos}\overset{\frown}{\theta} + \mathrm{i}\,\underset{\smile}{\sin}\overset{\frown}{\theta}\right)}{\overset{\frown}{\mathrm{rad}}} = \frac{\mathrm{i}\,\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,)}{\overset{\frown}{\mathrm{rad}}}$$

From this derivative, the authors derive a power series expansion for the complete exponential function:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (35) $\exp(\mathrm{i}\,\theta) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{2\pi\mathrm{i}\,\theta}{\Theta}\right)^n$

Which is clearer in dimensionally-annotated form:

(35a) $$\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\mathrm{i}\,\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^n$$

But the quotient here can be simplified:

(K22) $$\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}} = \frac{\theta\cdot\overset{\frown}{\mathrm{rad}}}{\overset{\frown}{\mathrm{rad}}} = \theta$$

So this is just the dimensionless angle in dimensionless radians. But this could be derived more succintly like so:

(K22a) $$\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}} = \underset{\smile}{\overset{\frown}{\theta}} = \theta$$

Which means that we can simplify the expression in Eq. (35a):

(35b) $$\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\mathrm{i}\,\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\mathrm{i}\,\underset{\smile}{\overset{\frown}{\theta}}\right)^n = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}\,\theta\,)^n$$

But we could get to this more directly and concisely even before doing the expanded sum:

(35c) $$\underset{\smile}{\exp}(\mathrm{i}\,\overset{\frown}{\theta}\,) = \exp\left(\mathrm{i}\,\underset{\smile}{\overset{\frown}{\theta}}\right) = \exp(\mathrm{i}\,\theta\,) = \sum_{n=0}^{\infty}\frac{1}{n!}(\mathrm{i}\,\theta\,)^n$$

From the power series in Eq. (35), the authors derive the exponential expression:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (36) $\exp(\mathrm{i}\,\theta) = e^{2\pi \mathrm{i}\theta/\Theta}$

Which again can be made clearer using annotations:

(36a) $\underset{\smile}{\exp}(\,\mathrm{i}\,\overset{\frown}{\theta}\,) = e^{\mathrm{i}\left(\overset{\frown}{\theta}/\overset{\frown}{\mathrm{rad}}\right)}$

But this can be expressed more succintly:

(36b) $\underset{\smile}{\exp}(\,\mathrm{i}\,\overset{\frown}{\theta}\,) = e^{\mathrm{i}\underset{\smile}{\overset{\frown}{\theta}}} = e^{\mathrm{i}\,\theta}$

In other words, we've shown the complete exponential function can be implemented by dividing the true·angle argument by a true·radian and using the dimensionless result as the exponent of a power of the natural base e.

Next, the authors break apart the power series for the complete exponential function into a power series for the complete cos function (the real part) and a power series for the complete sin function (the imaginary part):

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (37) $\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{2\pi\theta}{\Theta}\right)^{2n}$
>
> (38) $\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2\pi\theta}{\Theta}\right)^{2n+1}$

But we can make it clearer what is going on in both of these by applying dimensional annotations:

(37a) $\underset{\smile}{\cos}\left(\overset{\frown}{\theta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^{2n}$

(38a) $\underset{\smile}{\sin}\left(\overset{\frown}{\theta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^{2n+1}$

But then, just like in Eq. (35a), we can apply Eq. (K22a) to simplify and show that these equations are effectively stripping $\overset{\frown}{\theta}$ of its true·angle dimension and using θ in dimensionless-radians in the power series:

(37b) $\underset{\smile}{\cos}\left(\overset{\frown}{\theta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\underset{\smile}{\overset{\frown}{\theta}}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n}$

(38b) $\underset{\smile}{\sin}\left(\overset{\frown}{\theta}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{rad}}}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\underset{\smile}{\overset{\frown}{\theta}}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$

But, just like with Eq. (35b), we can get to this more directly and succinctly even before constructing the infinite sum:

$$(37c)\qquad \underset{\smile}{\cos}\left(\overset{\frown}{\theta}\right) = \cos\left(\underset{\smile}{\overset{\frown}{\theta}}\right) = \cos\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\theta^{2n}$$

$$(38c)\qquad \underset{\smile}{\sin}\left(\overset{\frown}{\theta}\right) = \sin\left(\underset{\smile}{\overset{\frown}{\theta}}\right) = \sin\theta = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\theta^{2n+1}$$

The authors now consider the natural logarithm, the inverse of exponentiation:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> If (39) $z = \exp(\mathrm{i}\theta)$ then the complete logarithmic function is
>
> $$(40)\qquad \log(z) = \mathrm{i}\theta + \mathrm{i}k\Theta$$
>
> where θ is the geometric angle with the dimension of angle and k is an integer.

We can express this more clearly by explicitly annotating the use of true·angle dimension:

$$(39a,40a)\quad z = \underset{\smile}{\exp}(\mathrm{i}\overset{\frown}{\theta}) \qquad \implies \qquad \overset{\frown}{\log}(z) = \mathrm{i}\overset{\frown}{\theta} + \mathrm{i}\overset{\frown}{\tau}k$$

Notice that the complete logarithm function $\overset{\frown}{\log}$ is annotated just like the complete inverse trionometric functions $\overset{\frown}{\arcsin}$ etc. In other words, it invokes the pure-math version of log, but in order to keep the equation dimensionally balanced, it must take the dimensionless result and multiply it by a true·radian to generate a true·angle result. This seems to answer a long-standing unanswered question that had been lingering for me from previous discussions about exponentiation and logarithms: Is a logarithm a kind of angle? At least for the cases of *imaginary* exponentiation we're encountering here, it appears to be.

Next, the authors use Eqs. (34) and (39) to calculate this derivative:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$(41)\qquad \frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{2\pi\mathrm{i}}{\Theta}z$$

This may be hard to see because the authors skip a couple steps. So let me spell it out, while applying dimensional annotations:

$$(41a)\qquad \frac{\mathrm{d}z}{\mathrm{d}\overset{\frown}{\theta}} = \underbrace{\frac{\mathrm{d}\,\underset{\smile}{\exp}\left(\mathrm{i}\overset{\frown}{\theta}\right)}{\mathrm{d}\overset{\frown}{\theta}}}_{1} = \underbrace{\frac{\mathrm{i}\,\underset{\smile}{\exp}(\,\mathrm{i}\,\overset{\frown}{\theta}\,)}{\overset{\frown}{\mathrm{rad}}}}_{2} = \underbrace{\frac{\mathrm{i}\,z}{\overset{\frown}{\mathrm{rad}}}}_{3} = \underbrace{\mathrm{i}\,z}_{4}$$

The steps are:

1. Replace z with the exponential from Eq. (40a).
2. Calculate derivative (apply Eq. (34a)).
3. Replace exponential with z from Eq. (40a).

4. Apply the radianic annotation to make it more succinct.

But what we really need is the reciprocal derivative:

$$\text{(41b)}\qquad \frac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}z} = \frac{1}{\mathrm{i}\,\underset{\smile}{z}} = \frac{\mathrm{i}}{\mathrm{i}}\cdot\frac{\overset{\frown}{1}}{\mathrm{i}\,z} = \frac{\mathrm{i}}{\mathrm{i}^2}\cdot\frac{\overset{\frown}{1}}{z} = \frac{\mathrm{i}}{-1}\cdot\frac{\overset{\frown}{1}}{z} = -\mathrm{i}\,\frac{\overset{\frown}{1}}{z} = -\mathrm{i}\,\frac{\overset{\frown}{\mathrm{rad}}}{z}$$

This just goes to show that $-\mathrm{i}$ is not only the *additive* inverse of i, but also its *multiplicative* inverse. This makes sense if you look at the unit circle on the complex plane: Multiplying any positive real number on the positive x-axis by i effectively rotates it counter-clockwise a quarter turn onto the positive y-axis. So it stands to reason that *dividing* by i would rotate it *clockwise* onto the *negative* y-axis.

With (41b) we can now get the derivative of the logarithm:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(42)}\qquad \frac{\mathrm{d}}{\mathrm{d}z}\log(z) = \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{i}\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{i}\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\Theta}{2\pi}\frac{1}{z}$$

Dimensionally annotated and broken down into steps:

$$\text{(42b)}\qquad \frac{\mathrm{d}}{\mathrm{d}z}\overset{\frown}{\log}(z) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{i}\,\overset{\frown}{\theta}}_{1} = \underbrace{\left(\frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\theta}}\mathrm{i}\,\overset{\frown}{\theta}\right)\frac{\mathrm{d}\overset{\frown}{\theta}}{\mathrm{d}z}}_{2} = \underbrace{\mathrm{i}\cdot\left(-\mathrm{i}\frac{\overset{\frown}{1}}{z}\right)}_{3} = \frac{\overset{\frown}{1}}{\underbrace{z}_{4}} = \frac{\overset{\frown}{\mathrm{rad}}}{z}$$

1. Substitute from Eq. (40a).
2. Apply the chain rule.
3. Resolve the trivial differential on the left, and substitute from Eq. (41b) on the right.
4. Simplify.

Next, the authors consider how to integrate the log function. Except they look ahead to a piece of the answer, and consider differentiating the product of z and $\log z$:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(43)}\qquad \frac{\mathrm{d}}{\mathrm{d}z}z\log(z) = \log(z) + \frac{\Theta}{2\pi} = \log(z) + \frac{\mathrm{d}}{\mathrm{d}z}\frac{\Theta}{2\pi}z$$

It's a little unclear how they got that first step, as well as what they're trying to say with that second step, so let's dimensionally annotate this and then break it down:

$$\text{(43a)}\qquad \frac{\mathrm{d}}{\mathrm{d}\,z}\left(z\cdot\overset{\frown}{\log}\,z\right) = \underbrace{\overset{\frown}{\log}\,z\cdot\frac{\mathrm{d}\,z}{\mathrm{d}\,z} + z\cdot\frac{\mathrm{d}\,\overset{\frown}{\log}\,z}{\mathrm{d}z}}_{1} = \overset{\frown}{\log}\,z\cdot 1 + \underbrace{z\cdot\frac{\overset{\frown}{1}}{z}}_{2} = \overset{\frown}{\log}\,z + \overset{\frown}{1} = \overset{\frown}{\log}\,z + \overset{\frown}{\mathrm{rad}}$$

1. Differentiate $z\cdot\overset{\frown}{\log}\,z$ by parts.

2. Apply Eq. (42a).

But that $\overset{\frown}{1}$ is a constant, so it can also be the result of differentiating $\overset{\frown}{z}$:

(43b) $$\frac{\mathrm{d}\overset{\frown}{z}}{\mathrm{d}z} = \overset{\frown}{1} \cdot \frac{\mathrm{d}z}{\mathrm{d}z} = \overset{\frown}{1}$$

Substituting Eq. (43b) into Eq. (43a):

(43c) $$\frac{\mathrm{d}}{\mathrm{d}z}\left(z \cdot \overset{\frown}{\log} z\right) = \overset{\frown}{\log} z + \frac{\mathrm{d}\overset{\frown}{z}}{\mathrm{d}z}$$

Rearranging:

(44a) $$\overset{\frown}{\log} z = \frac{\mathrm{d}}{\mathrm{d}z}\left(z \cdot \overset{\frown}{\log} z\right) - \frac{\mathrm{d}\overset{\frown}{z}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\left(z \cdot \overset{\frown}{\log} z - \overset{\frown}{z}\right)$$

Which is how the authors got to Eq. (44):

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (44) $$\frac{\mathrm{d}}{\mathrm{d}z}\left(z \log(z) - \frac{\Theta}{2\pi} z\right) = \log(z)$$

It's a straight shot from that to integrate both sides and get Eq. (45):

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (45) $$\int \mathrm{d}z \ \log(z) = z \ \log(z) - \frac{\Theta}{2\pi} z + \text{constant}$$

Dimensionally annotated:

(45a) $$\int \overset{\frown}{\log}(z)\,\mathrm{d}z = z\,\overset{\frown}{\log}(z) - \overset{\frown}{z} + \overset{\frown}{\theta}_0$$

Note that since this is an indefinite integration, we need to introduce an arbitrary constant in the solution, but to keep this equation dimensionally balanced, the constant must be commensurate and therefore must be a true·angle. Hence calling it $\overset{\frown}{\theta}_0$.

Next, the authors go back to the complete exponential function in Eq. (35) and consider extending it analytically from an imaginary argument i θ to a real argument ϕ:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (46) $$\exp(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{2\pi\phi}{\Theta}\right)^n$$

Dimensionally annotated:

(46a) $$\underset{\smile}{\exp}\,\overset{\frown}{\phi} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^n$$

But just like with Eq. (35a), we can simplify this:

$$\text{(46b)}\qquad \underset{\smile}{\exp}\,\overset{\frown}{\phi} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^{n} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\underset{\smile}{\overset{\frown}{\phi}}\right)^{n} = \sum_{n=0}^{\infty}\frac{1}{n!}\phi^{n}$$

But, as with Eq. (35b), we can get to this more directly and succintly before doing the expanded sum:

$$\text{(46c)}\qquad \underset{\smile}{\exp}\,\overset{\frown}{\phi} = e^{\underset{\smile}{\overset{\frown}{\phi}}} = e^{\phi} = \sum_{n=0}^{\infty}\frac{1}{n!}\phi^{n}$$

The authors use Eq. (46) to derive complete versions of the hyperbolic functions cosh and sinh:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$\text{(47)}\qquad \cosh(\phi) = \frac{\exp(\phi)+\exp(-\phi)}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{2\pi\phi}{\Theta}\right)^{2n}$$
>
> $$\text{(48)}\qquad \sinh(\phi) = \frac{\exp(\phi)-\exp(-\phi)}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{2\pi\phi}{\Theta}\right)^{2n+1}$$

Dimensionally annotated:

$$\text{(47a)}\qquad \underset{\smile}{\cosh}\,\overset{\frown}{\phi} = \frac{\underset{\smile}{\exp}\left(\overset{\frown}{\phi}\right)+\underset{\smile}{\exp}\left(-\overset{\frown}{\phi}\right)}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^{2n}$$

$$\text{(48a)}\qquad \underset{\smile}{\sinh}\,\overset{\frown}{\phi} = \frac{\underset{\smile}{\exp}\left(\overset{\frown}{\phi}\right)-\underset{\smile}{\exp}\left(-\overset{\frown}{\phi}\right)}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^{2n+1}$$

But just like for Eq. (35a), we can apply Eq. (K22a) to simplify:

$$\text{(47b)}\qquad \underset{\smile}{\cosh}\,\overset{\frown}{\phi} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^{2n} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\underset{\smile}{\overset{\frown}{\phi}}\right)^{2n} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\phi^{2n}$$

$$\text{(48b)}\qquad \underset{\smile}{\sinh}\,\overset{\frown}{\phi} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{\overset{\frown}{\phi}}{\overset{\frown}{\text{rad}}}\right)^{2n+1} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\underset{\smile}{\overset{\frown}{\phi}}\right)^{2n+1} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\phi^{2n+1}$$

But just like for Eq. (35b), we can get to this more directly and succintly even before doing the expanded sum:

$$\text{(47c)}\qquad \underset{\smile}{\cosh}\,\overset{\frown}{\phi} = \cosh\underset{\smile}{\overset{\frown}{\phi}} = \cosh\phi = \frac{e^{\phi}+e^{-\phi}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\phi^{2n}$$

$$\text{(48c)}\qquad \underset{\smile}{\sinh}\,\overset{\frown}{\phi} = \sinh\underset{\smile}{\overset{\frown}{\phi}} = \sinh\phi = \frac{e^{\phi}-e^{-\phi}}{2} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\phi^{2n+1}$$

C. Explicit Unit Expressions

In this section, the authors attempt to formalize how to handle alternative true·angle units, which was touched on earlier, in the **Units for Angles** section. Of course, I'd favor the same strategy I outlined there.

1. Arbitrary Units, $[\theta] = A$ $\quad\overset{\frown}{\Longrightarrow}\quad$ **1. Arbitrary true·angle units,** $\overset{\frown}{[\theta]} = \overset{\frown}{\text{A}}$

> **Mohr et al 1206_z (2022_d) wrote:**
>
> To connect to the earlier sections in which particular units are considered, the expressions in the previous section can be given for a particular, but arbitrary, choice of unit $[\theta] = \mathrm{A}$.

I think it's clearer if we annotate the choice of unit as a true·angle, rather than just another dimensionless angle unit: $[\overset{\frown}{\theta}] = \overset{\frown}{\mathrm{A}}$. (Note my change to the title of this subsection.)

> **Mohr et al 1206_z (2022_d) wrote:**
>
> In this case, we employ the relations $\Theta = \{\Theta\}_\mathrm{A}$ A ...

I'd dimensionally correct and annotate this relation, and avoid the clumsy brace syntax, and then rearrange to get another relation:

$$\text{(49a)} \qquad \overset{\frown}{\mathrm{tr}} = \overset{\frown}{\tau} = \tau \cdot \overset{\frown}{\mathrm{rad}} = \tau_\mathrm{A} \cdot \overset{\frown}{\mathrm{A}} \qquad \Longleftrightarrow \qquad \frac{\tau}{\tau_\mathrm{A}} = \frac{\overset{\frown}{\mathrm{A}}}{\overset{\frown}{\mathrm{rad}}}$$

So τ is the dimensionless number of true·radians in a true·turn; and τ_A is the dimensionless number of true·A's in a true·turn. But the ratio of τ to τ_A is the same as the ratio of a true·A to a true·radian. That's interesting…

> **Mohr et al 1206_z (2022_d) wrote:**
>
> …and $\theta = \{\theta\}_\mathrm{A}$ A ...

I'd also dimensionally correct and annotate this relation, and rearrange it as well:

$$\text{(49b)} \qquad \overset{\frown}{\theta} = \theta_\mathrm{A} \cdot \overset{\frown}{\mathrm{A}} \qquad \Longleftrightarrow \qquad \theta_\mathrm{A} = \frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{A}}}$$

> **Mohr et al 1206_z (2022_d) wrote:**
>
> …to write
>
> $$\text{(49)} \qquad \frac{\theta}{\Theta} = \frac{\{\theta\}_A}{\{\Theta\}_A}$$

Here's what that looks like after applying dimensional corrections:

$$\text{(49c)} \qquad \frac{\overset{\frown}{\theta}}{\overset{\frown}{\mathrm{tr}}} = \frac{\overset{\frown}{\theta}}{\overset{\frown}{\tau}} = \frac{\theta_\mathrm{A}}{\tau_\mathrm{A}}$$

Rather than write this relation, I'd prefer to break down $\overset{\frown}{\mathrm{A}}$ by true·radians, rearrange, and combine with Eq. (49a):

$$\text{(49d)} \qquad \overset{\frown}{\mathrm{A}} = \mathrm{A} \cdot \overset{\frown}{1} = \mathrm{A} \cdot \overset{\frown}{\mathrm{rad}} \qquad \Longleftrightarrow \qquad \mathrm{A} = \underset{\smile}{\overset{\frown}{\mathrm{A}}} = \frac{\overset{\frown}{\mathrm{A}}}{\overset{\frown}{1}} = \frac{\overset{\frown}{\mathrm{A}}}{\overset{\frown}{\mathrm{rad}}} = \frac{\tau}{\tau_\mathrm{A}}$$

Remember, A is the amount of dimensionless angle that a true·A unit ($\overset{\frown}{\mathrm{A}}$) is considered to have, both by "pure mathematics" as well as by SI. And that dimensionless angle is expressed in dimensionless radians. If you measure any true·angle $\overset{\frown}{\theta}$ in true·A units, ultimately that measure will need to be converted into dimensionless radians, before feeding that to the "pure math" version of any transcendental trigonometric, exponential, or hyperbolic function.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (50) $\exp(\mathrm{i}\,\theta) = b_{\mathrm{A}}{}^{\mathrm{i}\{\theta\}_{\mathrm{A}}}$
>
> where
>
> (51) $b_{\mathrm{A}} = e^{2\pi/\{\Theta\}_{\mathrm{A}}}$

Recasting this using my notation scheme:

(50a) $\underset{\smile}{\exp}(\mathrm{i}\,\widehat{\theta}) = b_{\mathrm{A}}{}^{\mathrm{i}\,\theta_{\mathrm{A}}}$

where

(51a) $b_{\mathrm{A}} = e^{\tau/\tau_{\mathrm{A}}} = e^{\mathrm{A}}$

So this new exponentiation base is just the base of the natural logarithms (e) raised to the number of true‧radians in a true‧A. That stands to reason, because if we plug this in to Eq. (50a), we get:

(50b) $\underset{\smile}{\exp}(\mathrm{i}\,\widehat{\theta}) = b_{\mathrm{A}}{}^{\mathrm{i}\,\theta_{\mathrm{A}}} = (e^{\mathrm{A}})^{\mathrm{i}\,\theta_{\mathrm{A}}} = e^{\mathrm{i}\,\theta_{\mathrm{A}}\cdot\mathrm{A}} = e^{\mathrm{i}\,\theta}$

In other words, we get the same result we would get if we first convert θ_{A} from dimensionless A units, to dimensionless radians, *before* raising e to that imaginary power.

But what happens if we take the log of b_{A}?

(51b) $\log(b_{\mathrm{A}}) = \log\left(e^{\tau/\tau_{\mathrm{A}}}\right) = \log\left(e^{\mathrm{A}}\right) = \mathrm{A}$

We get the number of true‧radians in a true‧A! So if we dimensionally correct this to use the complete function, we get:

(51c) $\widehat{\log}(b_{\mathrm{A}}) = \widehat{\mathrm{A}}$

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (52) $\log_{b_{\mathrm{A}}}(z) = \mathrm{i}\{\theta\}_{\mathrm{A}}$
>
> where
>
> (53) $z = b_{\mathrm{A}}{}^{\mathrm{i}\{\theta\}_{\mathrm{A}}}$

I'd rewrite this as:

(52a) $\log_{b_{\mathrm{A}}}(z) = \mathrm{i}\,\theta_{\mathrm{A}}$

where

(53a) $z = b_{\mathrm{A}}{}^{\mathrm{i}\,\theta_{\mathrm{A}}}$

But I'd derive the following from Eq. (52a):

(52b) $\log_{b_{\mathrm{A}}}(z) = \mathrm{i}\,\theta_{\mathrm{A}} = \frac{\mathrm{i}\,\theta}{\underbrace{\mathrm{A}}_{1}} = \frac{\log(z)}{\underbrace{\mathrm{A}}_{2}} = \frac{\widehat{\log}(z)}{\underbrace{\widehat{\mathrm{A}}}_{3}}$

Steps:

1. Substitute from Eq. (49b).
2. Substitute from Eq. (40).
3. Dimensionally correct top and bottom.

So $\log_{b_{\mathrm{A}}}$ just gives you the same result as using the complete version of $\overset{\frown}{\log}$ to get a true·angle, and then breaking that true·angle down by true·A units ($\overset{\frown}{\mathrm{A}}$). The question is, if we already have a complete function $\overset{\frown}{\log}$, why would we need a custom function $\log_{b_{\mathrm{A}}}$ for our arbitrary unit, much less multiple custom functions for multiple units?

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (54) $\qquad \cos_{\mathrm{A}}\left(\{\theta\}_{\mathrm{A}}\right) = \dfrac{{b_{\mathrm{A}}}^{\mathrm{i}\{\theta\}_{\mathrm{A}}} + {b_{\mathrm{A}}}^{-\mathrm{i}\{\theta\}_{\mathrm{A}}}}{2}$
>
> (55) $\qquad \sin_{\mathrm{A}}\left(\{\theta\}_{\mathrm{A}}\right) = \dfrac{{b_{\mathrm{A}}}^{\mathrm{i}\{\theta\}_{\mathrm{A}}} - {b_{\mathrm{A}}}^{-\mathrm{i}\{\theta\}_{\mathrm{A}}}}{2\mathrm{i}}$

I'd rewrite this as:

(54a) $\qquad \cos_{\mathrm{A}}\left(\theta_{\mathrm{A}}\right) = \dfrac{{b_{\mathrm{A}}}^{\mathrm{i}\,\theta_{\mathrm{A}}} + {b_{\mathrm{A}}}^{-\mathrm{i}\,\theta_{\mathrm{A}}}}{2}$

(55a) $\qquad \sin_{\mathrm{A}}\left(\theta_{\mathrm{A}}\right) = \dfrac{{b_{\mathrm{A}}}^{\mathrm{i}\,\theta_{\mathrm{A}}} - {b_{\mathrm{A}}}^{-\mathrm{i}\,\theta_{\mathrm{A}}}}{2\,\mathrm{i}}$

But of course $\theta_{\mathrm{A}} \cdot \overset{\frown}{\mathrm{A}} = \overset{\frown}{\theta}$, so:

(54b) $\qquad \cos_{\mathrm{A}}(\theta_{\mathrm{A}}) = \underset{\smile}{\cos}(\,\theta_{\mathrm{A}} \cdot \overset{\frown}{\mathrm{A}}\,) = \underset{\smile}{\cos}(\,\overset{\frown}{\theta}\,) = \cos(\,\underset{\smile}{\overset{\frown}{\theta}}\,) = \cos(\theta) = \dfrac{e^{\mathrm{i}\,\theta} + e^{-\mathrm{i}\,\theta}}{2}$

(55b) $\qquad \sin_{\mathrm{A}}(\theta_{\mathrm{A}}) = \underset{\smile}{\sin}(\,\theta_{\mathrm{A}} \cdot \overset{\frown}{\mathrm{A}}\,) = \underset{\smile}{\sin}(\,\overset{\frown}{\theta}\,) = \sin(\,\underset{\smile}{\overset{\frown}{\theta}}\,) = \sin(\theta) = \dfrac{e^{\mathrm{i}\,\theta} - e^{-\mathrm{i}\,\theta}}{2\,\mathrm{i}}$

So again, if we already have complete versions of $\overset{\frown}{\sin}$ and $\overset{\frown}{\cos}$, we can get the right results by first taking our dimensionless number of a custom unit and converting that to a true·angle, by multiplying by the custom unit itself as a true·angle. So why should we proliferate multiple custom versions of all our trig functions?

---

2. Radian unit, $\mathrm{A} = \mathrm{rad}$ $\qquad \overset{\frown}{\Longrightarrow} \qquad$ **2. True·radian unit**, $\overset{\frown}{\mathrm{A}} = \overset{\frown}{\mathrm{rad}}$

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> If A is the radian, we have $2\pi/\{\Theta\}_{\mathrm{rad}} = 1$, $b_{\mathrm{rad}} = e$, and
>
> (56) $\qquad \exp\left(\mathrm{i}\,\theta\right) = e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}}$

If the true·angle unit $\overset{\frown}{\mathrm{A}}$ is the true·radian, we have $\mathrm{A} = 1$, $b_{\mathrm{A}} = e$, and we already have θ meaning the dimensionless number of true·radians in true·angle $\overset{\frown}{\theta}$. So:

(56a) $\qquad \underset{\smile}{\exp}\left(\mathrm{i}\overset{\frown}{\theta}\right) = e^{\mathrm{i}\,\underset{\smile}{\overset{\frown}{\theta}}} = e^{\mathrm{i}\,\theta}$

> **Mohr et al 1206_z (2022_d) wrote:**
>
> If (57) $z = e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}}$, then
>
> (58) $\log_e(z) = \ln(z) = \mathrm{i}\{\theta\}_{\mathrm{rad}}$

(58a) $\widehat{\log}_e(z) = \widehat{\ln}(z) = \mathrm{i}\,\widehat{\theta}$

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (59) $\cos_{\mathrm{rad}}(\{\theta\}_{\mathrm{rad}}) = \dfrac{e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}} + e^{-\mathrm{i}\{\theta\}_{\mathrm{rad}}}}{2}$
>
> (60) $\sin_{\mathrm{rad}}(\{\theta\}_{\mathrm{rad}}) = \dfrac{e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}} - e^{-\mathrm{i}\{\theta\}_{\mathrm{rad}}}}{2\,\mathrm{i}}$

Of course, this is just:

(59a) $$\cos_{\mathrm{rad}}(\{\theta\}_{\mathrm{rad}}) = \frac{e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}} + e^{-\mathrm{i}\{\theta\}_{\mathrm{rad}}}}{2}$$

(60a) $$\sin_{\mathrm{rad}}(\{\theta\}_{\mathrm{rad}}) = \frac{e^{\mathrm{i}\{\theta\}_{\mathrm{rad}}} - e^{-\mathrm{i}\{\theta\}_{\mathrm{rad}}}}{2\,\mathrm{i}}$$

---

3. Revolution or cycle unit, $\mathrm{A} = \mathrm{rev}$ $\quad \widehat{\Longrightarrow} \quad$ **3. True·turn unit**, $\widehat{\mathrm{A}} = \widehat{\mathrm{tr}}$

> **Mohr et al 1206_z (2022_d) wrote:**
>
> In this case, $2\pi/\{\Theta\}_{\mathrm{rev}} = 2\pi$, $b_{\mathrm{rev}} = e^{2\pi}$, and
>
> (61) $\exp(\mathrm{i}\,\theta) = e^{2\pi\,\mathrm{i}\{\theta\}_{\mathrm{rev}}}$

In this case, the true·angle unit $\widehat{\mathrm{A}}$ is the true·turn. Let's symbolize that $\widehat{\mathrm{A}} = \widehat{\odot} = \widehat{\tau}$. Then we have

(61a)
$$\begin{aligned}
&\mathrm{A} = \odot = \tau \\
&b_{\mathrm{A}} = b_{\odot} = e^{\tau} \\
&\theta = \theta_{\mathrm{A}}\mathrm{A} = \theta_{\odot}\cdot\odot = \theta_{\odot}\cdot\tau \\
&\underset{\smile}{\exp}(\,\mathrm{i}\,\widehat{\theta}\,) = e^{\mathrm{i}\,\theta_{\odot}\cdot\tau} = e^{\mathrm{i}\,\theta}
\end{aligned}$$

There's no way around this: Once we get down to the underlying pure math versions of transcendental functions, if we use anything but dimensionless radians we're forced to throw in an extra factor, in this case τ. This was already generalized above for arbitrary units:

(50b) $$\underset{\smile}{\exp}(\mathrm{i}\,\widehat{\theta}) = \cdots = e^{\mathrm{i}\,\theta_{\mathrm{A}}\cdot\mathrm{A}} = e^{\mathrm{i}\,\theta}$$

Applications

In this section, the authors consider some problematic cases where granting angle true dimensionality at first leads to apparent dimensional non-homogeneity, but we can see how to resolve these cases by going back to first principles and applying fundamentals of calculus.

A. Centripetal Acceleration

This subsection considers centripetal acceleration:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (66) $$a_c = \frac{v^2}{r} = r\omega^2$$

Let's recast this in dimensionally corrected and annotated form:

(66a) $$a_c = \frac{v^2}{r} = r\overset{\frown}{\omega}^2$$

where:

a_c = **centripetal acceleration**
Dimension: acceleration = length / time^2 ✓

v = **tangential velocity**
Dimension: velocity = length / time

$\frac{v^2}{r}$
Dimension: velocity^2 / length = (length^2/time^2) / length = length/time^2 = acceleration ✓

This is dimensionally homogeneous with a_c.

r = radius Dimension: length

$\overset{\frown}{\omega}$ = angular velocity Dimension: angular·velocity = true·angle / time

$r\overset{\frown}{\omega}^2$
Dimension: length · angular·velocity^2 = length · (true·angle / time)^2 = length · true·angle^2 / time^2 ✗

This is NOT dimensionally homogeneous with a_c. How do we resolve this contradiction, without abandoning true dimensionality for angles? In their previous article (Mohr & Phillips $11ƐƐ_z$ (2015_d), https://arxiv.org/pdf/1409.2794),[2] the authors tried to imply (in a table) that the radius r here was actually a radiality $\underset{\smile}{r}$. This would cancel out one of these true·angle dimensions, but it left centripetal acceleration with a remaining true·angle dimension that cannot really be justified. And that still would not balance here. This time, however, they go back to first principles and apply calculus to reason about what is really going on here.

First, they spell out the parametric vector equation for the position of an object undergoing uniform circular motion, in terms of its cylindrical coordinates:

[2]See my review of this article, at https://dozenal.org/article-volan-review-mohr-phillips-11BB.

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (67) $$\mathbf{r}(r,\phi) = r\cos(\phi)\,\hat{\mathbf{i}} + r\sin(\phi)\,\hat{\mathbf{j}}$$
>
> where $\phi = \omega t$.

Here is the equivalent, dimensionally corrected and annotated:

(67a) $$\mathbf{r}(r,\overset{\frown}{\phi}) = r\underset{\smile}{\cos}(\overset{\frown}{\phi})\,\hat{\mathbf{i}} + r\underset{\smile}{\sin}(\overset{\frown}{\phi})\,\hat{\mathbf{j}}$$

where

$\mathbf{r}(r,\overset{\frown}{\phi})$ = **position** vector in terms of cylindrical coordinates
Dimension: length

$\overset{\frown}{\phi} = \overset{\frown}{\omega} t$ = **phase** true·angle
Dimension: true·angle = angular·velocity · time = (true·angle / time) · time

$\overset{\frown}{\omega} = \dfrac{\mathrm{d}\overset{\frown}{\phi}}{\mathrm{d}t}$ = **angular velocity** as time differential of phase true·angle
Dimension: angular·velocity = true·angle / time

$\hat{\mathbf{i}}, \hat{\mathbf{j}}$ = **unit vectors** along coordinate axes in plane of rotation
Dimension: 1 (dimensionless)

Then the authors take the second derivative of this with respect to time to get the centripetal acceleration:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> The centripetal acceleration is
>
> (68) $$\mathbf{a}_{\mathrm{c}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}(r,\phi) = \omega^2 \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\mathbf{r}(r,\phi)$$

Let's look at what they did here in dimensionally corrected and annotated form, and then try to fill in how they got that result:

(68a) $$\mathbf{a}_{\mathrm{c}} = \frac{\mathrm{d}^2\,\mathbf{r}(r,\overset{\frown}{\phi})}{\mathrm{d}t^2} = \underbrace{\frac{\mathrm{d}^2\,\mathbf{r}(r,\overset{\frown}{\phi})}{\mathrm{d}\overset{\frown}{\phi}{}^2} \cdot \frac{\mathrm{d}\overset{\frown}{\phi}{}^2}{\mathrm{d}t^2}}_{1} = \underbrace{\frac{\mathrm{d}^2\,\mathbf{r}(r,\overset{\frown}{\phi})}{\mathrm{d}\overset{\frown}{\phi}{}^2} \cdot \overset{\frown}{\omega}{}^2}_{2}$$

Steps:

1. Apply the chain rule, to separate out how the position vector changes with respect to phase true·angle, from how the phase true·angle changes with respect to time.
2. The first derivative of phase true·angle with respect to time is angular velocity. So we get the square of that here.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> From Eqs. (25), (29), and (32), we have
>
> (69) $$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\cos(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \cos(\phi)$$
>
> (70) $$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\sin(\phi) = -\left(\frac{2\pi}{\Theta}\right)^2 \sin(\phi)$$

We've seen this before. Here it is in dimensionally corrected and annotated form:

(69a) $$\frac{\mathrm{d}^2 \underset{\smile}{\cos}\,(\overset{\frown}{\phi})}{\mathrm{d}\overset{\frown}{\phi}{}^2} = -\frac{\underset{\smile}{\cos}\,(\overset{\frown}{\phi})}{\overset{\frown}{\mathrm{rad}}{}^2}$$

(70a) $$\frac{\mathrm{d}^2 \underset{\smile}{\sin}\,(\overset{\frown}{\phi})}{\mathrm{d}\overset{\frown}{\phi}{}^2} = -\frac{\underset{\smile}{\sin}\,(\overset{\frown}{\phi})}{\overset{\frown}{\mathrm{rad}}{}^2}$$

Combining these with a factor of r and then attaching them to unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, the second derivative actually reproduces the original function, but with two powers of true·radian extracted out into the denominator (along with a minus sign):

(69a+70a) $$\frac{\mathrm{d}^2\, \mathbf{r}\,(r, \overset{\frown}{\phi})}{\mathrm{d}\overset{\frown}{\phi}{}^2} = -\frac{r\,\underset{\smile}{\cos}\,(\overset{\frown}{\phi})\,\hat{\mathbf{i}} + r\,\underset{\smile}{\sin}\,(\overset{\frown}{\phi})\,\hat{\mathbf{j}}}{\overset{\frown}{\mathrm{rad}}{}^2} = -\frac{\mathbf{r}\,(r, \overset{\frown}{\phi})}{\overset{\frown}{\mathrm{rad}}{}^2}$$

Substituting this into Eq. (68) gives us:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (71) $$\mathbf{a}_{\mathrm{c}} = -\left(\frac{2\pi\omega}{\Theta}\right)^2 \mathbf{r}\,(r, \phi)$$

Dimensionally corrected and annotated:

(71a) $$\mathbf{a}_{\mathrm{c}} = -\frac{\mathbf{r}\,(r, \overset{\frown}{\phi}) \cdot \overset{\frown}{\omega}{}^2}{\overset{\frown}{\mathrm{rad}}{}^2}$$

The authors also take the norm of this to put this into scalar form:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (72) $$a_{\mathrm{c}} = r\left(\frac{2\pi\omega}{\Theta}\right)^2$$

Which we can do in annotated form:

(72a) $$a_{\mathrm{c}} = \frac{r\,\overset{\frown}{\omega}{}^2}{\overset{\frown}{\mathrm{rad}}{}^2}$$

Dimension: length · angular·velocity^2 / true·angle^2 = length · (true·angle^2 / time^2) / true·angle^2 = length / time^2 = acceleration ✓

This is the complete equation for centripetal acceleration with respect to radius and angular velocity. And we see that it is dimensionally balanced.

I am not explicitly grouping those true·radian factors with the angular·velocity factors, but here the authors choose to do so. They go on to exploit that to cancel out the true·angle factor in the angular·velocity:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (73) $\quad a_c = r\omega_{rad}{}^2$

Which would look like this in my notation:

(73a) $$a_c = r\left(\frac{\overset{\frown}{\omega}}{\overset{\frown}{\text{rad}}}\right)^2 = r\left(\underset{\smile}{\overset{\frown}{\omega}}\right)^2 = r\omega^2$$

But I think this is misleading, because it implies that the true·angle dimension in angular velocity is somehow optional when it gets in the way. That pair of true·radian factors came from a different differential than the pair of angular velocity factors, so I'd rather just keep them separate. Both pairs came from applying the chain rule, so they were guaranteed to balance each other out, but they are two separate pieces of the puzzle.

Plus, without this explicit true·angle dimension, angular velocity would be indistinguishable from frequency. The whole point of this article is to assert angularity as a true dimension in order to avoid that, and particularly to avoid the errors that come from confusing radians per time with turns per time.

Something to underline here is the fact that the true·angle factors that get pulled out by differentiation here are specifically true·*radians*. It does not matter what units we choose to measure rotation with here, we would still get true·radians as the factors. But this does reinforce the fact that the true·radian constitutes the most natural choice for a *coherent* unit of true·angle.

B. Volume Integration

The next example application the article tackles is doing integral calculus over a volume. Since this involves multiple geometric dimensions (3D) as well as multiple variables for the coordinates, this can be handled via multivariable/multidimensional calculus.

The authors start with an infinitesimal element of volume expressed in Cartesian coordinates (x, y, z), and consider how the same volume element may be expressed in spherical coordinates $(r, \overset{\frown}{\theta}, \overset{\frown}{\phi})$:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (74) $\quad \langle \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \rangle = \mathsf{L}^3$
>
> (75) $\quad \langle r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \rangle = \mathsf{L}^3\,\mathsf{A}^2$

Here, angle brackets are used as an operator which yields the dimensionality of the enclosed expression. Let's dimensionally correct and annotate the angular coordinates:

(74a) $\quad \langle \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \rangle = \text{length}^3 = \text{volume}$ ✓

(75a) $\quad \left\langle r^2\,\mathrm{d}r\,\underset{\smile}{\sin}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi} \right\rangle = \text{length}^3\,\text{true·angle}^2$ ✗

We'll see in a moment how they derived Eq. (75). But note the dilemma here: If we give the angular coordinates true·angle dimension like this, we get an apparent contradiction about the dimensionality of the volume element. How do we resolve this? We can't just posit that the radius r here is actually a radiality $\underset{\smile}{r}$, because there are only two true·angle variables in this formula, but there are three instances of r. That stands to reason, because we're trying cube length to get a volume. Giving each r radiality dimension would divide by one too many true·angle dimensions. We need to go back to first principles and figure out what's actually going on here.

First, what's the relationship between the Cartesian and spherical coordinates? The authors show how the Cartesian coordinates can be expressed as functions of the spherical coordinates:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (76) $$x = r\ \sin(\theta)\ \cos(\phi)$$
>
> (77) $$y = r\ \sin(\theta)\ \sin(\phi)$$
>
> (78) $$z = r\ \cos(\theta)$$

Let's make dimensional corrections and annotate:

(76a) $$x = r\,\underset{\smile}{\sin}(\overset{\frown}{\theta})\,\underset{\smile}{\cos}(\overset{\frown}{\phi})$$

(77a) $$y = r\,\underset{\smile}{\sin}(\overset{\frown}{\theta})\,\underset{\smile}{\sin}(\overset{\frown}{\phi})$$

(78a) $$z = r\,\underset{\smile}{\cos}(\overset{\frown}{\theta})$$

Next, how do we transform an infinitessimal volume element from a differential in Cartesian coordinates into a differential in spherical coordinates? The authors invoke something known as a Jacobian determinant:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (79) $$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \cdots$$

Let's dimensionally correct and annotate:

(79a) $$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \overset{\frown}{\theta}} & \frac{\partial x}{\partial \overset{\frown}{\phi}} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \overset{\frown}{\theta}} & \frac{\partial y}{\partial \overset{\frown}{\phi}} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \overset{\frown}{\theta}} & \frac{\partial z}{\partial \overset{\frown}{\phi}} \end{vmatrix} \mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi} = \cdots$$

The 3x3 grid here represents the Jacobian determinant, which does the transformation. Each cell contains a partial derivative, which differentiates one of the Cartesian coordinates with respect to one of the spherical coordinates. A partial derivative treats the coordinate in its denominator as a variable, and holds any expressions involving the other coordinates to be constants. The authors substitute the spherical-coordinate functions from Eqs. (76-78) for each of the Cartesian coordinates and then calculate the partial derivatives. Let's step through that:

Let's calculate each partial derivative individually. First, let's differentiate each Cartesian coordinate by r (**radius**), treating $\overset{\frown}{\theta}$ and $\overset{\frown}{\phi}$ as constants:

(79a.1) $$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi})\right) = \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi})$$

(79a.2) $$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi})\right) = \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi})$$

(79a.3) $$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}\left(r \underset{\smile}{\cos}(\overset{\frown}{\theta})\right) = \underset{\smile}{\cos}(\overset{\frown}{\theta})$$

Next, let's differentiate each Cartesian coordinate by $\overset{\frown}{\theta}$ (**inclination**), treating r and $\overset{\frown}{\phi}$ as constants:

(79a.4) $$\frac{\partial x}{\partial \overset{\frown}{\theta}} = \frac{\partial}{\partial \overset{\frown}{\theta}}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi})\right) = r\,\frac{\underset{\smile}{\cos}(\overset{\frown}{\theta})}{\overset{\frown}{\mathrm{rad}}}\,\underset{\smile}{\cos}(\overset{\frown}{\phi}) = r \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1}$$

(79a.5) $$\frac{\partial y}{\partial \overset{\frown}{\theta}} = \frac{\partial}{\partial \overset{\frown}{\theta}}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi})\right) = r\,\frac{\underset{\smile}{\cos}(\overset{\frown}{\theta})}{\overset{\frown}{\mathrm{rad}}}\,\underset{\smile}{\sin}(\overset{\frown}{\phi}) = r \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1}$$

(79a.6) $$\frac{\partial z}{\partial \overset{\frown}{\theta}} = \frac{\partial}{\partial \overset{\frown}{\theta}}\left(r \underset{\smile}{\cos}(\overset{\frown}{\theta})\right) = -r\,\frac{\underset{\smile}{\sin}(\overset{\frown}{\theta})}{\overset{\frown}{\mathrm{rad}}} = -r \underset{\smile}{\sin}(\overset{\frown}{\theta}) \cdot \overset{\frown}{\mathrm{rad}}^{-1}$$

Now let's differentiate each Cartesian coordinate by $\overset{\frown}{\phi}$ (**azimuth**), treating r and $\overset{\frown}{\theta}$ as constants:

(79a.7) $$\frac{\partial x}{\partial \overset{\frown}{\phi}} = \frac{\partial}{\partial \overset{\frown}{\phi}}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi})\right) = -r \underset{\smile}{\sin}(\overset{\frown}{\theta})\,\frac{\underset{\smile}{\sin}(\overset{\frown}{\phi})}{\overset{\frown}{\mathrm{rad}}} = -r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1}$$

(79a.8) $$\frac{\partial y}{\partial \overset{\frown}{\phi}} = \frac{\partial}{\partial \overset{\frown}{\phi}}\left(r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi})\right) = r \underset{\smile}{\sin}(\overset{\frown}{\theta})\,\frac{\underset{\smile}{\cos}(\overset{\frown}{\phi})}{\overset{\frown}{\mathrm{rad}}} = r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1}$$

(79a.9) $$\frac{\partial z}{\partial \overset{\frown}{\phi}} = \frac{\partial}{\partial \overset{\frown}{\phi}}\left(r \underset{\smile}{\cos}(\overset{\frown}{\theta})\right) = 0$$

Finally, let's assemble the results into the matrix:

(79b) $$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \begin{vmatrix} \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) & r \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1} & -r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1} \\ \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) & r \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1} & r \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \cdot \overset{\frown}{\mathrm{rad}}^{-1} \\ \underset{\smile}{\cos}(\overset{\frown}{\theta}) & -r \underset{\smile}{\sin}(\overset{\frown}{\theta}) \cdot \overset{\frown}{\mathrm{rad}}^{-1} & 0 \end{vmatrix} \mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi} = \cdots$$

We now need to calculate the determinant. In general, a determinant is calculated as follows:

(79b.1) $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - hf) + b(fg - id) + c(dh - ge) = aei - ahf + bfg - bid + cdh - cge$$

But before they proceeded with this, apparently the authors noticed that each of these terms would end up with two powers of $\overset{\frown}{\mathrm{rad}}^{-1}$ (unless the term winds up being 0). So they went ahead and factored those out of the determinant — except they represented them using their unnecessarily complex and obfuscating formula $\dfrac{2\pi}{\Theta}$:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$\text{(79)}\quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \left(\frac{2\pi}{\Theta}\right)^2 \begin{vmatrix} \sin(\theta)\cos(\phi) & r\,\cos(\theta)\cos(\phi) & r\,\sin(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) & r\,\cos(\theta)\sin(\phi) & r\,\sin(\theta)\cos(\phi) \\ \cos(\theta) & -r\,\sin(\theta) & 0 \end{vmatrix} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Let's do the same (but more plainly):

$$\text{(79c)}\qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \overset{\frown}{\mathrm{rad}}^{-2} \begin{vmatrix} \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) & r\underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) & -r\underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \\ \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) & r\underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) & r\underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \\ \underset{\smile}{\cos}(\overset{\frown}{\theta}) & -r\underset{\smile}{\sin}(\overset{\frown}{\theta}) & 0 \end{vmatrix} \mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi}$$

Each of the terms is also going to wind up with two powers of r, so let's factor those out as well:

$$\text{(79d)}\qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \overset{\frown}{\mathrm{rad}}^{-2}r^2 \begin{vmatrix} \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) & \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) & -\underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) \\ \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) & \underset{\smile}{\cos}(\overset{\frown}{\theta})\underset{\smile}{\sin}(\overset{\frown}{\phi}) & \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}(\overset{\frown}{\phi}) \\ \underset{\smile}{\cos}(\overset{\frown}{\theta}) & -\underset{\smile}{\sin}(\overset{\frown}{\theta}) & 0 \end{vmatrix} \mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi}$$

The remaining determinant now only contains trig functions. The authors calculate that out to yield:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$\text{(79)}\qquad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \left(\frac{2\pi}{\Theta}\right)^2 r^2\,\sin(\theta)\;\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Wait, what? That whole determinant just equals $\underset{\smile}{\sin}(\overset{\frown}{\theta})$? Let's see how they got that. First, let's expand out the determinant following the formula in Eq. (79b.1), dropping any 0 terms:

$$\text{(79d.1)}\qquad \begin{aligned} &\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \\ &\overset{\frown}{\mathrm{rad}}^{-2}r^2\left(\underset{\smile}{\sin}^3(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\phi}) + \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\phi}) + \underset{\smile}{\sin}^3(\overset{\frown}{\theta})\underset{\smile}{\sin}^2(\overset{\frown}{\phi}) + \underset{\smile}{\sin}(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\theta})\underset{\smile}{\sin}^2(\overset{\frown}{\phi})\right)\mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi} \end{aligned}$$

We can see that each of the terms has at least one power of $\underset{\smile}{\sin}(\overset{\frown}{\theta})$, so let's factor that out:

$$\text{(79d.2)}\qquad \begin{aligned} &\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \cdots = \\ &\overset{\frown}{\mathrm{rad}}^{-2}r^2\underset{\smile}{\sin}(\overset{\frown}{\theta})\left(\underset{\smile}{\sin}^2(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\phi}) + \underset{\smile}{\cos}^2(\overset{\frown}{\theta})\underset{\smile}{\cos}^2(\overset{\frown}{\phi}) + \underset{\smile}{\sin}^2(\overset{\frown}{\theta})\underset{\smile}{\sin}^2(\overset{\frown}{\phi}) + \underset{\smile}{\cos}^2(\overset{\frown}{\theta})\underset{\smile}{\sin}^2(\overset{\frown}{\phi})\right)\mathrm{d}r\,\mathrm{d}\overset{\frown}{\theta}\,\mathrm{d}\overset{\frown}{\phi} \end{aligned}$$

Let's apply the distributive law to get:

$$\text{d}x\,\text{d}y\,\text{d}z = \cdots = \quad (79\text{d}.3)$$
$$\widehat{\text{rad}}^{-2} r^2 \sin(\widehat{\theta}) \left[\left(\sin^2(\widehat{\theta}) + \cos^2(\widehat{\theta}) \right) \cdot \cos^2(\widehat{\phi}) + \left(\sin^2(\widehat{\theta}) + \cos^2(\widehat{\theta}) \right) \cdot \sin^2(\widehat{\phi}) \right] \text{d}r\,\text{d}\widehat{\theta}\,\text{d}\widehat{\phi}$$

Let's apply the distributive law again:

$$(79\text{d}.4) \quad \text{d}x\,\text{d}y\,\text{d}z = \cdots = \widehat{\text{rad}}^{-2} r^2 \sin(\widehat{\theta}) \left(\sin^2(\widehat{\theta}) + \cos^2(\widehat{\theta}) \right) \cdot \left(\sin^2(\widehat{\phi}) + \cos^2(\widehat{\phi}) \right) \text{d}r\,\text{d}\widehat{\theta}\,\text{d}\widehat{\phi}$$

We now have a product of two expressions that each are the sum of the squares of the sine and cosine of a given angle. For any angle, such an sum of squares equals 1. So we can cancel them out to yield:

$$(79\text{e}) \quad \text{d}x\,\text{d}y\,\text{d}z = \cdots = \widehat{\text{rad}}^{-2} r^2 \sin(\widehat{\theta})\,\text{d}r\,\text{d}\widehat{\theta}\,\text{d}\widehat{\phi}$$

This is the complete formula for the transformation. We now can see that $< \widehat{\text{rad}}^{-2} r^2 \sin(\widehat{\theta})\,\text{d}r\,\text{d}\widehat{\theta}\,\text{d}\widehat{\phi} > = \text{length}^3 = \text{volume}$.

The $\widehat{\text{rad}}^{-2}$ factor we got from the partial differentials in the Jacobian determinant cancel out the true·angle dimensions in $\text{d}\widehat{\theta}$ and $\text{d}\widehat{\phi}$. This resolves the confict.

B. Water Waves

The next application the authors consider is calculating the **phase velocity** c_{p} of a water wave. Note that this is not actually the velocity of the *water*, but of the *wave*. When a wave propagates across the surface of the water, the individual water molecules only move *vertically*, but the *disturbance* in the water gets transmitted *horizontally*. This is why the term here is *phase* velocity, because what this velocity is tracking is how any given phase of the wave is moving. Whether you watch the wave crests or the wave troughs, or any point in between, this velocity will be the same.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> $$(85) \quad c_{\text{p}} = \frac{\omega}{k} = \frac{\sqrt{gk\tanh(kh)}}{k} = \sqrt{gh}\sqrt{\frac{\tanh(kh)}{kh}} \quad \textbf{?}$$

where:

c_{p} = **phase·velocity** of the wave
Dimension: $\langle c_{\text{p}} \rangle = \text{length} \cdot \text{time}^{-1}$ ✓

$\widehat{\omega}$ = angular·velocity of the wave (although Mohr et al call it the "frequency" of the wave)
Dimension: $\left\langle \widehat{\omega} \right\rangle = \text{true·angle} \cdot \text{time}^{-1}$

$\widehat{k} = \dfrac{1}{\underset{\smile}{\lambda\!\!\!^{-}}}$ = wave·curvature of the wave (although conventionally termed the "wave number")
Dimension: $\left\langle \widehat{k} \right\rangle = \text{true·angle} \cdot \text{length}^{-1}$

$\underset{\smile}{\lambda\!\!\!^{-}}$ = wave·radiality of the wave (although conventionally called the "reduced wave length").
Dimension: $\left\langle \underset{\smile}{\lambda\!\!\!^{-}} \right\rangle = \text{length} \cdot \text{true·angle}^{-1}$

g = acceleration due to gravity
Dimension: $\langle g \rangle = \text{length} \cdot \text{time}^{-2} = \text{acceleration}$

h = depth of the water
Dimension: $\langle h \rangle = \text{length}$

The first part of the equation has the expected dimensionality:

$$\frac{\overset{\frown}{\omega}}{\overset{\frown}{k}} = \overset{\frown}{\omega} \cdot \underset{\smile}{ƛ}$$

Dimension: $\left\langle \frac{\overset{\frown}{\omega}}{\overset{\frown}{k}} \right\rangle = \left\langle \overset{\frown}{\omega} \cdot \underset{\smile}{ƛ} \right\rangle = (\text{true·angle} \cdot \text{time}^{-1}) \cdot (\text{length} \cdot \text{true·angle}^{-1}) = \text{length} \cdot \text{time}^{-1}$ ✓

But looking at the latter part of the equation, we get a contradictory dimensionality (hence the red question mark). To make this easier to see, the authors square the formula and simplify, to express the relationship as:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (86) $\omega^2 = gk \tanh(kh)$

Let's apply annotations to make the dimensional corrections and the complete function explicit:

(86a) $\overset{\frown}{\omega}^2 = g\overset{\frown}{k}\,\underset{\smile}{\tanh}(\overset{\frown}{k}h)$ **?**

The left hand side of the equation has the expected dimension:

Dimension: $\left\langle \overset{\frown}{\omega}^2 \right\rangle = \text{true·angle}^2 \cdot \text{time}^{-2}$ ✓

As for the right hand side:

Dimension: $\left\langle g\overset{\frown}{k}\,\underset{\smile}{\tanh}(\overset{\frown}{k}h) \right\rangle = \cdots$

First, the argument of the hyperbolic tangent here is the product of wave radiality and height, so it has true·angle dimension:

Dimension: $\left\langle \overset{\frown}{k}h \right\rangle = (\text{true·angle} \cdot \text{length}^{-1}) \cdot \text{length} = \text{true·angle}$

This is the right dimension for an argument to the complete $\underset{\smile}{\tanh}$ function. This function first reduces its argument to dimensionless radians, and then invokes the "pure math" version of tanh. But the result of this is dimensionless, so it can simply be dropped from the overall dimension calculation:

Dimension: $\left\langle g\overset{\frown}{k}\,\underset{\smile}{\tanh}(\overset{\frown}{k}h) \right\rangle = \left\langle g\overset{\frown}{k} \right\rangle = (\text{length} \cdot \text{time}^{-2}) \cdot (\text{true·angle} \cdot \text{length}^{-1}) = \text{true·angle} \cdot \text{time}^{-2}$ ✗

The result here has one less true·angle dimension than expected, so something is wrong with this formula. We need to go back to square one and derive this formula via calculus, asserting true angular dimension from the start.

To that end, the authors consider a quantity that many people may never have heard of before: so called **velocity·potential** (ϕ). They give no context on what this is, so let me fill that in. Despite using the Greek letter ϕ, this is not an angle, but rather a scalar line integral of the **velocity vector v**, as long as the water flow is **irrotational** (no rotations or vortices).

(86b) $\phi(\mathbf{r}) = \phi(\mathbf{r}_0) + \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v} \cdot d\boldsymbol{\ell}$

Dimension: $\langle \phi \rangle = \text{velocity} \cdot \text{length} = \text{length}^2 \cdot \text{time}^{-1}$

Or, flipping this around, the velocity vector is the *gradient* of ϕ:

(86c) $\nabla\phi = \mathbf{v}$

That upside-down delta is called a "nabla", or the "del" operator. This is an important differential operator for vector and multivariable calculus. It represents a partial differential performed on each Cartesian coordinate in Euclidean 2D or 3D space, multiplying each by the corresponding unit vector along its coordinate axis, to form a new vector quantity:

(86d) $$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$$

For surface waves on water, we can recast velocity potential as a scalar-valued differentiable function of several variables: $\phi(t, x, z)$. Note that we only care about the x (horizontal) component (direction in which the wave propagates) and the z (vertical) component (the direction that the water molecules actually move) but not the y (transverse) component. Applying the del operator to it yields the 2-dimensional gradient of ϕ, which in this case is the velocity vector.

(86e) $$\nabla\phi(t, x, z) = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z} = \mathbf{v}$$

From this, the authors set up a second-order differential equation involving ϕ:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (87) $$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z = 0$$
>
> with the boundary condition
>
> (88) $$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = 0$$

Dimension: $\left\langle\frac{\partial\phi}{\partial z}\right\rangle = (\text{length}^2 \cdot \text{time}^{-1}) \cdot \text{length}^{-1} = \text{length} \cdot \text{time}^{-1} = \text{velocity}$

Dimension: $\left\langle g\frac{\partial\phi}{\partial z}\right\rangle = (\text{length} \cdot \text{time}^{-2}) \cdot (\text{length} \cdot \text{time}^{-1}) = \text{length}^2 \cdot \text{time}^{-3}$

Dimension: $\left\langle\frac{\partial^2\phi}{\partial t^2}\right\rangle = (\text{length}^2 \cdot \text{time}^{-1}) \cdot \text{time}^{-2} = \text{length}^2 \cdot \text{time}^{-3}$

Solving the differential equation:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> The solution is
>
> (89) $$\phi = \frac{\cosh k(z+h)}{k \sinh kh}\omega a \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)$$
>
> where a is a normalization constant and $k_z = 0$.

Let's annotate the dimensional corrections and the complete functions:

(89a) $$\phi = \frac{\underbrace{\cosh\overset{\frown}{k}(z+h)}}{\underbrace{\overset{\frown}{k}\sinh\overset{\frown}{k}h}}\overset{\frown}{\omega}a\underbrace{\sin(\overset{\frown}{\mathbf{k}} \cdot \mathbf{x} - \overset{\frown}{\omega}t)}$$

Calling a a "normalization constant" is accurate, but not very enlightening. In fact, a is the **wave amplitude**, i.e., half the height of the waves from trough to crest, or the maximum deviation of the water from flat stillness. This has the dimension $\langle a \rangle = \text{length}$.

If we take the second time derivative of ϕ, we wind up reconstituting ϕ again, but with a minus sign, and with two extra factors of $\text{true·radian}^{-1} = \overset{\frown}{1}^{-1} = \underset{\smile}{1}$, as well as two factors of $\overset{\frown}{\omega} = \frac{\partial \overset{\frown}{\theta}}{\partial t}$ (angular·velocity), pulled out from differentiating the $\underset{\smile}{\sin}(\cdots - \overset{\frown}{\omega} t)$ function, as we've seen in other examples.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> Differentiation gives
>
> (90) $$\frac{\partial^2 \phi}{\partial t^2} = -\left(\frac{2\pi\omega}{\Theta}\right)^2 \phi$$

Annotating the dimensional corrections:

(90a) $$\frac{\partial^2 \phi}{\partial t^2} = -\phi \cdot \underset{\smile}{1}^2 \cdot \overset{\frown}{\omega}^2$$

Plugging into Eq. (86), this gives us:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (91) $$\omega^2 = \frac{g}{\phi}\left(\frac{\Theta}{2\pi}\right)^2 \frac{\partial \phi}{\partial z}\bigg|_{z=0} = gk\frac{\Theta}{2\pi} \tanh kh$$

(91a) $$\overset{\frown}{\omega}^2 = \frac{g}{\phi} \cdot \overset{\frown}{1}^2 \cdot \frac{\partial \phi}{\partial z}\bigg|_{z=0} = g\overset{\frown}{k} \cdot \overset{\frown}{1} \cdot \underset{\smile}{\tanh} \overset{\frown}{kh}$$

Dimension: $\left\langle g\overset{\frown}{k} \cdot \overset{\frown}{1} \cdot \underset{\smile}{\tanh} \overset{\frown}{kh} \right\rangle = (\text{length} \cdot \text{time}^{-2}) \cdot (\text{true·angle} \cdot \text{length}^{-1}) \cdot \text{true·angle} = \text{true·angle}^2 \cdot \text{time}^{-2}$ ✓

This now has the expected dimension, resolving the conflict.

D. Units for the Cyclotron Resonance Frequency

The next example the authors consider is the resonant frequency of a cyclotron. A cyclotron is a device which uses a magnetic field in a flat cylindrical vacuum chamber to accelerate charged particles along a spiral path, due to the Lorentz force. The cylindrical geometry allows the particles to encounter the acceleration region repeatedly, so they can accumulate a lot of energy. The resonant frequency is the angular velocity needed in order to get the particles into a circular orbit within the field and accumulate energy most efficiently. This depends on the strength of the magnetic field, as well as the charge and mass of the particles. This example explores the intersection between angular mechanics and electromagnetism.

Before we look at that, let's get a little background. What is a "field"? A field is any kind of phenomenon which can be described as having a (different) value at every point in space. Some fields are scalar fields, meaning the "value" at each point is some scalar quantity without any specified directionality. Other fields are vector fields, meaning the "value" is some vector quantity with a magnitude and a direction. A "force field" is a vector field where the relevant quantity is a force, caused by some phenomenon.

A "magnetic field" is a force vector field, where the cause is some amount of magnetism. What exactly is "magnetism"? If you take some quantity of "electricity" q (by which we mean some amount of static electric charge), and give it a velocity $\mathbf{v}$in your inertial reference frame, then you get a quantity of "magnetism" $q\mathbf{v}$. Magnetism is effectively "electricity-in-motion".

A quantity of *stationary* electricity generates an *electric* field **E**. This field causes forces on other quantities of stationary electricity, either attracting them if they have opposite sign, or repelling them if they have the same sign. But a quantity of *magnetism* (electricity-in-motion) generates a *magnetic* field **B**. This causes forces on other quantities of *magnetism* (electricity-in-motion). A magnetic field is effectively what an electric field looks like, when observed from a moving inertial reference frame; as such, it is affected by the consequences of Special Relativity.

At the end of the thread "Maxwell, Metavariables, Radiels, etc", starting on post #114, I proposed an overhaul of the terminology of electromagnetism. One of the things I suggested was to interpret the **-ic** syllable in the words **electric** and **magnetic** in terms of the ISO-31 standard for reciprocal quantities. So **electric** would mean "per quantity of electricity"; **magnetic** would mean "per quantity of magnetism".

Hence, an "**electric force**" field **E** would be measured in units of **force**, per unit of **electricity**. A "**magnetic force**" field **B** would be measured in units of **force**, per unit of **magnetism**.

Since a quantity of electricity is a scalar, the force vector imparted on it by an electric force field would have either the same direction, or the opposite direction, as the direction of the field, depending on the sign of q:

(K23) $\mathbf{F} = q\mathbf{E}$

But a quantity of magnetism is a vector, pointing in the direction of motion. So the force imparted on it by a magnetic field would be the *cross-product* of the magnetism vector with the magnetic field vector:

(K24) $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$

That means that the force vector would be pointing *perpendicular* to the plane containing the magnetism vector $q\mathbf{v}$ and the magnetic field vector **B**. A particle experiencing a force perpendicular to its motion will tend to get deflected into a curved or even circular path.

With that in mind, let's look at the equation the authors quote for the resonant frequency:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (92) $\omega = \dfrac{qB_c}{m}$

Annotating the dimensional corrections for angular measure:

(92a) $\overset{\frown}{\omega} = \dfrac{qB_c}{m}$ **?**

where

$\overset{\frown}{\omega}$ is the resonant **angular·velocity** (angular "frequency")
Dimension: $\left\langle \overset{\frown}{\omega} \right\rangle = \text{true·angle} \cdot \text{time}^{-1}$ ✓

q = electric charge ("quantity of (static) **electricity**") of a particle
Dimension: $\langle q \rangle = \text{electricity}$

m = **mass** of a particle
Dimension: $\langle m \rangle = \text{mass}$

B_c = "classical magnetic field" = **magnetic·force** = $\|\mathbf{B}_c\|$
Dimension: $\langle B_c \rangle = \text{force} \cdot \text{magnetism}^{-1} = (\text{mass} \cdot \text{acceleration}) \cdot (\text{electricity} \cdot \text{velocity})^{-1} = \text{mass} \cdot \text{time}^{-1} \cdot \text{electricity}^{-1}$

$\dfrac{qB_c}{m}$
Dimension: $\left\langle \dfrac{qB_c}{m} \right\rangle = \text{electricity} \cdot \text{mass}^{-1} \cdot (\text{mass} \cdot \text{time}^{-1} \cdot \text{electricity}^{-1}) = \text{time}^{-1}$ ✗

So the left hand side of the equation is calling for an angular dimension which seems to be missing from the right hand side. How do we resolve this?

The authors assume that the particle in question at resonant frequency will be moving in a circular path, so they consider its position vector function $\mathbf{x}(t)$ in terms of cylindrical coordinates:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (93) $\mathbf{x} = r\cos(\omega t)\hat{\mathbf{i}} + r\sin(\omega t)\hat{\mathbf{j}}$

Annotating the dimensional corrections and complete functions:

(93a) $\mathbf{x}(t) = r\underset{\smile}{\cos}(\overset{\frown}{\omega}t)\hat{\mathbf{i}} + r\underset{\smile}{\sin}(\overset{\frown}{\omega}t)\hat{\mathbf{j}}$

where

$\mathbf{x}(t)$ = position vector function
Dimension: $\langle \mathbf{x} \rangle = \text{length}$

r = radius of rotation
Dimension: $\langle r \rangle = \text{length}$

t = time
Dimension: $\langle t \rangle = \text{time}$

$\overset{\frown}{\theta} = \overset{\frown}{\omega}t$
Dimension: $\left\langle \overset{\frown}{\omega}t \right\rangle = (\text{true·angle} \cdot \text{time}^{-1}) \cdot \text{time} = \text{true·angle}$

Dimension: $\left\langle \underset{\smile}{\cos}(\overset{\frown}{\omega}t) \right\rangle = \left\langle \underset{\smile}{\sin}(\overset{\frown}{\omega}t) \right\rangle = 1 = (\text{Dimensionless})$

$\hat{\mathbf{i}}, \hat{\mathbf{j}}$ = unit vectors along Cartesian coordinate axes in plane of rotation
Dimension: $\left\langle \hat{\mathbf{i}} \right\rangle = \left\langle \hat{\mathbf{j}} \right\rangle = 1 = (\text{Dimensionless})$

They then proceed to take the derivative of this with respect to time to get the velocity vector function $\mathbf{v}(t)$:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (94) $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left[r\cos(\omega t)\hat{\mathbf{i}} + r\sin(\omega t)\hat{\mathbf{j}}\right] = \left(\frac{2\pi\omega}{\Theta}\right)\left[-r\sin(\omega t)\hat{\mathbf{i}} + r\cos(\omega t)\hat{\mathbf{j}}\right]$

Let's break this down a bit to see how they got this. First, they need to apply the chain rule, taking into account that the complete functions and true·angles need to be converted to pure math forms. This then requires the trig functions to be differentiated in terms of dimensionless $\mathrm{d}\theta = \mathrm{d}(\omega t)$, which means pulling a true·angle constant out of the differential $\mathrm{d}\overset{\frown}{\theta} = \mathrm{d}(\overset{\frown}{\omega}t)$. In the end, the pure math forms are converted back to complete functions and true□angles:

(94a) $\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = -\left(\frac{2\pi\omega}{\Theta}\right)^2\left[r\cos(\omega t)\hat{\mathbf{i}} + r\sin(\omega t)\hat{\mathbf{j}}\right] = -\left(\frac{2\pi\omega}{\Theta}\right)^2\mathbf{x}$

They then take the second derivative to get the acceleration vector function $\mathbf{a}(t)$:

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> (96) $$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \frac{\mathrm{d}}{\mathrm{d}t}\left[r\cos(\omega t)\hat{\mathbf{i}} + r\sin(\omega t)\hat{\mathbf{j}}\right] = \left(\frac{2\pi\omega}{\Theta}\right)\left[-r\sin(\omega t)\hat{\mathbf{i}} + r\cos(\omega t)\hat{\mathbf{j}}\right]$$

(96a) $$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\left[r\underset{\smile}{\cos}(\overset{\frown}{\omega}t)\hat{\mathbf{i}} + r\underset{\smile}{\sin}(\overset{\frown}{\omega}t)\hat{\mathbf{j}}\right]\cdot \underset{\smile}{1}^2 \cdot \overset{\frown}{\omega}^2 = -\mathbf{x}\cdot \underset{\smile}{1}^2 \cdot \overset{\frown}{\omega}^2$$

In the process of doing this, we see that the acceleration function becomes a simple function of the original position function. But now we see that not only have two factors of $\overset{\frown}{\omega}$ been pulled out, but also two factors of $\underset{\smile}{1} = \overset{\frown}{\mathrm{rad}}^{-1}$ to balance.

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> For a magnetic field given by $\mathbf{B}_\mathrm{c} = -B_\mathrm{c}\hat{\mathbf{k}}$, the classical force is
>
> (98) $$\mathbf{F}_\mathrm{c} = q\mathbf{v}\times\mathbf{B}_\mathrm{c} = -qB_\mathrm{c}\frac{2\pi\omega}{\Theta}\left[r\sin(\omega t)\hat{\mathbf{j}} + r\cos(\omega t)\hat{\mathbf{i}}\right] = -qB_\mathrm{c}\frac{2\pi\omega}{\Theta}\mathbf{x}$$

(98a) $$\mathbf{F}_\mathrm{c} = q\mathbf{v}\times\mathbf{B}_\mathrm{c} = -qB_\mathrm{c}\cdot\left[r\underset{\smile}{\cos}(\overset{\frown}{\omega}t)\hat{\mathbf{i}} + r\underset{\smile}{\sin}(\overset{\frown}{\omega}t)\hat{\mathbf{j}}\right]\cdot\underset{\smile}{1}\cdot\overset{\frown}{\omega} = -qB_\mathrm{c}\cdot\mathbf{x}\cdot\underset{\smile}{1}\cdot\overset{\frown}{\omega}$$

> **Mohr et al 1206$_z$ (2022$_d$) wrote:**
>
> Thus from Newton's law, $\mathbf{F}_\mathrm{c} = m\mathbf{a}$, we have
>
> (100) $$qB_\mathrm{c} = m\frac{2\pi\omega}{\Theta}$$
>
> or
>
> (101) $$\omega = \frac{qB_\mathrm{c}}{m}\frac{\Theta}{2\pi}$$
>
> This is the complete equation, and the dimensions, AT^{-1}, match.
>
> In SI units, the frequency is
>
> (102) $$\omega = \left\{\frac{qB_\mathrm{c}}{m}\right\}\times\begin{cases}\mathrm{rad\,s^{-1}} \\ \dfrac{\mathrm{cycles\ s^{-1}}}{2\pi} = \dfrac{\mathrm{Hz}}{2\pi}\end{cases}$$

It might be clearer if we use quantitels for the **international** system units:

(102a🌐) $$\overset{\frown}{\omega} = \{\overset{\frown}{\omega}\}_{🌐}\cdot[\overset{\frown}{\omega}]_{🌐}$$

(102b🌐) $$\{\overset{\frown}{\omega}\}_{🌐} = \omega_{🌐} = \frac{q_{🌐}\,B_{\mathrm{c}🌐}}{m_{🌐}}$$

(102c🌐) $$[\overset{\frown}{\omega}]_{🌐} = 🌐\measuredangle\mathrm{vc}\ell = \begin{cases}\overset{\frown}{1}\cdot 🌐\mathrm{tm}\ell^{-1} = \overset{\frown}{\mathrm{rad}}\cdot\mathrm{s}^{-1} \\ \dfrac{\overset{\frown}{\tau}}{\tau}\cdot 🌐\mathrm{tm}\ell^{-1} = \dfrac{\overset{\frown}{\mathrm{tr}}\cdot\mathrm{s}^{-1}}{\tau} = \dfrac{\overset{\frown}{\mathrm{cycle}}\cdot\mathrm{s}^{-1}}{\tau} = \dfrac{\overset{\odot}{\mathrm{Hz}}}{\tau}\end{cases}$$

But we could also do this using any other system of units:

$$(102a\boxdot)\quad \overset{\frown}{\omega} = \{\overset{\frown}{\omega}\}_{\boxdot} \cdot [\overset{\frown}{\omega}]_{\boxdot}$$

$$(102b\boxdot)\quad \{\overset{\frown}{\omega}\}_{\boxdot} = \omega_{\boxdot} = \frac{q_{\boxdot}\, B_{c\boxdot}}{m_{\boxdot}}$$

$$(102c\boxdot)\quad [\overset{\frown}{\omega}]_{\boxdot} = \boxdot\measuredangle\mathrm{vc}\ell = \begin{cases} \overset{\frown}{1} \cdot \boxdot\mathrm{tm}\ell^{-1} = \overset{\frown}{\mathrm{rad}} \cdot \boxdot\mathrm{tm}\ell^{-1} \\ \dfrac{\overset{\frown}{\tau}}{\tau} \cdot \boxdot\mathrm{tm}\ell{-1} = \dfrac{\overset{\frown}{\mathrm{tr}} \cdot \boxdot\mathrm{tm}\ell^{-1}}{\tau} \end{cases}$$

But both $\overset{\odot}{\mathrm{Hz}}$ and $\overset{\frown}{\mathrm{tr}}$ are non-coherent units, hence the emergence of τ as an "extraneous" factor.

E. Classical Pendulum

The next example the authors consider is a classical pendulum, which is an example of a simple harmonic oscillator. They make the usual simplifying assumption of a pendulum oscillating back and forth over a small angle, which lets us exploit the approximation of $\sin\theta \approx \theta$ (when expressed in radians). The classical formula for the "angular frequency" of oscillation is:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$(103)\qquad \omega = \sqrt{\frac{g}{L}}$$

This is dimensionally balanced under SI's assumption that angles are dimensionless ratios of lengths, but as with all the other examples, the authors imply dimensionally correcting this to give angles first-class dimensionality. If we annotate this equation to make this clear, we see that it is no longer dimensionally balanced:

$$(103b)\quad \overset{\frown}{\omega} = \sqrt{\frac{g}{L}} \qquad \textbf{?}$$

where $\overset{\frown}{\omega}$ = angular velocity of the pendulum
Dimension: $\langle\overset{\frown}{\omega}\rangle = \mathsf{angular{\cdot}velocity} = \mathsf{true{\cdot}angle} \cdot \mathsf{time}^{-1}$ ✓

g = acceleration due to gravity
Dimension: $\langle g\rangle = \mathsf{acceleration} = \mathsf{length} \cdot \mathsf{time}^{-2}$

L = length of the pendulum arm
Dimension: $\langle L\rangle = \mathsf{length}$

So:
Dimension: $\left\langle\sqrt{\frac{g}{L}}\right\rangle = \sqrt{\frac{\mathsf{length}\cdot\mathsf{time}^{-2}}{\mathsf{length}}} = \sqrt{\mathsf{time}^{-2}} = \mathsf{time}^{-1}$ ✗

Once again, we have a flawed formula whose derivation has apparently overlooked something, due to SI's treatment of angles. So we must go back to first principles and re-derive this while being more rigorous about our dimensional analysis. To that end, Mohr et al provide Figure 2 depicting the pendulum and the forces that act on it:

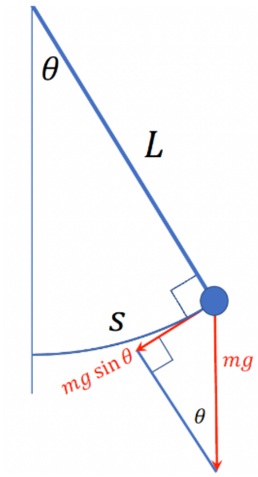

FIG. 2. Classical pendulum

where:

m = the mass of the pendulum bob

$\theta(t)$ = the angle by which the pendulum bob has deviated from vertical at time t, dimensionally corrected as true□angle $\overset{\frown}{\theta}(t)$

$s(t)$ = the arc length by which the pendulum bob has deviated from vertical at time t

The full force of gravity acting on the pendulum bob is of course $F_g = -mg$, however, a component of this force is canceled out by the tension in the direction of the pendulum arm, leaving only the component in the orthogonal direction to accelerate the bob: $F = -mg\sin(\theta(t))$, dimensionally corrected as $F = -mg\underset{\smile}{\sin}\left(\overset{\frown}{\theta}(t)\right)$. The net acceleration a can be described as the second-order time differential of deviation of the arc length:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (104) $$m\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2} = -mg\sin(\theta(t))$$

Annotating the implied dimensional corrections gives us:

(104a) $$m\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2} = -mg\underset{\smile}{\sin}(\overset{\frown}{\theta}(t))$$

The authors cancel out mass m from both sides. Then they avail themselves of the approximation for small angles:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (105) $$\sin(\theta(t)) = \frac{2\pi\theta(t)}{\Theta} + \cdots$$

But once again, they've obfuscated what this means by using the formula $\frac{\Theta}{2\pi}$ to represent a constant equal to one true□radian.

Annotations let us see this more clearly:

$$\text{(105a)} \quad \underset{\smile}{\sin}\left(\overset{\frown}{\theta}(t)\right) = \sin\left(\underset{\smile}{\overset{\frown}{\theta}}(t)\right) = \sin\left(\frac{\overset{\frown}{\theta}(t)}{\overset{\frown}{\text{rad}}}\right) \approx \frac{\overset{\frown}{\theta}(t)}{\overset{\frown}{\text{rad}}} = \underset{\smile}{1}\cdot\overset{\frown}{\theta}(t)$$

But we can also look at this as allowing the complete version of the sine function and the true angle to revert to pure-math equivalents:

$$\text{(105b)} \quad \underset{\smile}{\sin}\left(\overset{\frown}{\theta}(t)\right) = \sin\left(\underset{\smile}{\overset{\frown}{\theta}}(t)\right) = \sin\left(\theta(t)\right) \approx \theta(t)$$

Next, the authors apply Eq. (26). Although I skipped over it before, it amounted to this, when combined with Eq. (29):

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(26+29)} \quad \mathcal{C} = \frac{2\pi}{\Theta} = \frac{1}{r}\frac{\mathrm{d}s}{\mathrm{d}\theta}$$

Substituting $r = L$ as the "radius" of the pendulum, and then rearranging and setting up integration, they derive the formula:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(106)} \quad \frac{2\pi}{\Theta}\int \mathrm{d}\theta = \frac{1}{L}\int \mathrm{d}s$$

In terms of our annotation scheme, this is equivalent to:

$$\text{(106a)} \quad \underset{\smile}{1}\int \mathrm{d}\overset{\frown}{\theta} = \frac{1}{\overset{\frown}{\text{rad}}}\int \mathrm{d}\overset{\frown}{\theta} = \frac{1}{L}\int \mathrm{d}s$$

Evaluating the integrals and then rearranging yields:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(107)} \quad s(t) = \frac{2\pi L}{\Theta}\theta(t)$$

The equivalent under the annotation scheme is:

$$\text{(107a)} \quad s(t) = \frac{L}{\overset{\frown}{\text{rad}}}\overset{\frown}{\theta}(t) = \underset{\smile}{L}\,\overset{\frown}{\theta}(t)$$

Note that $\underset{\smile}{L}$ is effectively the "radiality" of the pendulum. Thus we've reproduced the fundamental relationship between arc length s, angle $\overset{\frown}{\theta}$, and radiality $\underset{\smile}{L}$.

Combining Eq. (104) with Eq. (105) and Eq. (107), then rearranging, the authors derive:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> $$\text{(108)} \quad \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + \frac{g}{L}\theta(t) = 0$$

In our annotation scheme, this is equivalent to:

(108a) $$\frac{\mathrm{d}^2\overset{\frown}{\theta}(t)}{\mathrm{d}t^2} + \frac{g}{L}\overset{\frown}{\theta}(t) = 0$$

The authors quote the general solution for this as:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (109) $$\theta(t) = a\sin(\omega t) + b\cos(\omega t)$$

Rendered in our annotation scheme, this is:

(109a) $$\overset{\frown}{\theta}(t) = \overset{\frown}{a}\underset{\smile}{\sin}(\overset{\frown}{\omega}t) + \overset{\frown}{b}\underset{\smile}{\cos}(\overset{\frown}{\omega}t)$$

Differentiating this twice with respect to time, as in other example applications, pulls out two factors of angular velocity, as well as two factors of their constant $\mathcal{C}$:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (110) $$\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} = -\left(\frac{2\pi\omega}{\Theta}\right)^2\theta(t)$$

In terms of our annotation scheme, this pulls out two factors of $\overset{\frown}{\omega}$ as well as two true·radianic factors $\underset{\smile}{1}$:

(110a) $$\frac{\mathrm{d}^2\overset{\frown}{\theta}(t)}{\mathrm{d}t^2} = -\overset{\frown}{\theta}(t)\cdot\underset{\smile}{1}^2\cdot\overset{\frown}{\omega}^2 = -\overset{\frown}{\theta}(t)\cdot\overset{\frown}{\mathrm{rad}}^{-2}\cdot\overset{\frown}{\omega}^2$$

Combining with Eq. (108) and rearranging yields:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (111) $$\omega = \sqrt{\frac{g}{L}}\frac{\Theta}{2\pi}$$

Rendered in our annotation scheme reveals the true·radian factor missing from Eq. (103):

(111a) $$\overset{\frown}{\omega} = \sqrt{\frac{g}{L}}\overset{\frown}{1} = \sqrt{\frac{g}{L}}\overset{\frown}{\mathrm{rad}}$$

This now balances the dimensionality of the equation:

(111b) $$\left\langle\overset{\frown}{\omega}\right\rangle = \left\langle\sqrt{\frac{g}{L}}\overset{\frown}{\mathrm{rad}}\right\rangle = \text{true·angle}\cdot\text{time}^{-1} = \text{angular·velocity}$$

Once again, going back to fundamentals and then applying calculus techniques rigorously resolves an apparent dimensional discrepancy.

F. Jacobi Elliptic Functions

While the trigonometric functions are defined in terms of the unit circle, the Jacobi elliptical functions are a generalization to the other conic sections, particularly the ellipse. Rather than one angular parameter θ, the Jacobi functions are defined to take two parameters, u and k:

- u is called the "amplitude parameter" and is a generalization of angle, analogous to θ. Like θ, it is conventionally treated as dimensionless, but ought to be a true·angle, annotated as $\overset{\frown}{u}$.
- k is called the "elliptical modulus" and represents the degree of deviation of the Jacobi function from trigonometric behavior. It is a dimensionless parameter in the range $0 \leq k \leq 1$.

When $k = 0$, the Jacobi functions default to trigonometric behavior, as Mohr et al point out:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (113) $\quad \mathrm{sn}\,(u, 0) = \sin u$
>
> (114) $\quad \mathrm{cn}\,(u, 0) = \cos u$

We can turn these into complete functions in terms of true·angles, using our annotation scheme:

(113a) $\quad \underset{\smile}{\mathrm{sn}}\,(\overset{\frown}{u}, 0) = \underset{\smile}{\sin}\,\overset{\frown}{u}$

(114a) $\quad \underset{\smile}{\mathrm{cn}}\,(\overset{\frown}{u}, 0) = \underset{\smile}{\cos}\,\overset{\frown}{u}$

Although Mohr et al don't mention this, at the opposite extreme when $k = 1$, the Jacobi functions default to hyperbolic behavior:

(113b) $\quad \underset{\smile}{\mathrm{sn}}\,(\overset{\frown}{u}, 1) = \underset{\smile}{\tanh}\,\overset{\frown}{u}$

(114b) $\quad \underset{\smile}{\mathrm{cn}}\,(\overset{\frown}{u}, 1) = \underset{\smile}{\mathrm{sech}}\,\overset{\frown}{u}$

The Jacobi functions are defined as "reciprocals" of the following **elliptical integral**, which Mohr et al cite:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (115) $\quad u = \int_0^{\phi} \frac{d\phi\prime}{\sqrt{1 - k^2 \sin^2 \phi\prime}}$

Let's annotate our dimensional corrections:

(115a) $\quad \overset{\frown}{u} = \int_0^{\overset{\frown}{\phi}} \frac{d\overset{\frown}{\theta}}{\sqrt{1 - k^2 \underset{\smile}{\sin}^2 \overset{\frown}{\theta}}}$

This equation relates the amplitude parameter $\overset{\frown}{u}$ to an angle $\overset{\frown}{\phi}$ known as the **amplitude**. A special **amplitude function (am)** is usually defined for this:

(115b) $\quad \overset{\frown}{\phi} = \underset{\smile}{\overset{\frown}{\mathrm{am}}}\,(\overset{\frown}{u}, k)$

Note that my dimensional correction of this places both a true·radian operator and a true·radianic operator on the **am** function, which is a situation we haven't encountered before. These operators can coexist on this function without canceling each other out. The true·radian operator on **am** is needed to balance the true·radian operator on ϕ. The true·radianic operator is needed to make **am** a complete function that can thereby take $\overparen{u}$ as a parameter. The proper way to cancel out the operators is as follows:

(115c) $\overparen{\phi} = \underparen{\overparen{\mathrm{am}}}\,(\overparen{u}, k) \underset{1}{\Longleftrightarrow} \underparen{\overparen{\phi}} = \underparen{\mathrm{am}}\,(\overparen{u}, k) \underset{2}{\Longleftrightarrow} \underparen{\overparen{\phi}} = \mathrm{am}\,(\underparen{\overparen{u}}, k) \underset{3}{\Longleftrightarrow} \phi = \mathrm{am}\,(u, k)$

1. Move the true·radian operator on **am** to the opposite side of the equation to become a true·radianic operator on ϕ.
2. Move the true·radianic operator on **am** onto its u parameter.
3. Operators on ϕ and u cancel out.

With ϕ thus defined, Mohr et al can now define the **sn** and **cn** functions in terms of ϕ:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (116) $\mathrm{sn}\,(u, k) = \sin\phi$
>
> (117) $\mathrm{cn}\,(u, k) = \cos\phi$

We can dimensionally correct and annotate these, and then substitute Eq. (115b) into them:

(116a) $\underparen{\mathrm{sn}}\,(\overparen{u}, k) = \underparen{\sin}\,\overparen{\phi} = \underparen{\sin}\,(\underparen{\overparen{\mathrm{am}}}\,(\overparen{u}, k))$

(117a) $\underparen{\mathrm{cn}}\,(\overparen{u}, k) = \underparen{\cos}\,\overparen{\phi} = \underparen{\cos}\,(\underparen{\overparen{\mathrm{am}}}\,(\overparen{u}, k))$

Mohr et al also define function **dn**, known as the **delta amplitude** function. This characterizes the *distortion factor* introduced by k:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> (118) $\mathrm{dn}\,(u, k) = \sqrt{1 - k^2 \sin^2 \phi}$

We can dimensionally correct and annotate this:

(118a) $\underparen{\mathrm{dn}}\,(\overparen{u}, k) = \sqrt{1 - k^2 \underparen{\sin}^2 \overparen{\phi}}$

Next, Mohr et al consider taking Eq. (115) and differentiating it with respect to ϕ:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> From Eq. (115), we have
>
> (119) $\dfrac{\mathrm{d}u}{\mathrm{d}\phi} = \dfrac{1}{\sqrt{1 - k^2 \sin^2 \phi}}$

We can dimensionally correct and annotate this:

(119a) $$\frac{\mathrm{d}\overset{\frown}{u}}{\mathrm{d}\overset{\frown}{\phi}} = \frac{1}{\sqrt{1 - k^2 \underset{\smile}{\sin}^2 \overset{\frown}{\phi}}}$$

This gives the rate of change of $\overset{\frown}{u}$ with respect to $\overset{\frown}{\phi}$. We can easily take the reciprocal of that, giving us the rate of change of $\overset{\frown}{\phi}$ with respect to $\overset{\frown}{u}$. But we notice that the result is the same as the **delta amplitude** function (**dn**):

(119b) $$\frac{\mathrm{d}\overset{\frown}{\phi}}{\mathrm{d}\overset{\frown}{u}} = \sqrt{1 - k^2 \underset{\smile}{\sin}^2 \overset{\frown}{\phi}} = \underset{\smile}{\mathrm{dn}}\,(\overset{\frown}{u}, k)$$

We can make use of this when we differentiate other Jacobi functions:

> **Mohr et al 1206_z (2022_d) wrote:**
>
> and so
>
> (120) $$\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{sn}\,(u,k) = \frac{\mathrm{d}}{\mathrm{d}u}\sin\phi = \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\sin\phi\right)\frac{\mathrm{d}\phi}{\mathrm{d}u} \quad = \frac{2\pi}{\Theta}\cos\phi\sqrt{1 - k^2\sin^2\phi} = \frac{2\pi}{\Theta}\mathrm{cn}\,(u,k)\,\mathrm{dn}\,(u,k)$$

Once again, Mohr et al have pulled a radianic factor out of the differential $\frac{\mathrm{d}}{\mathrm{d}u}$, but have expressed it with their overly complicated and obscure formula $\frac{2\pi}{\Theta}$. With our dimensional correcting annotations, we can express it more simply as $\underset{\smile}{1}$:

(120a) $$\frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{u}}\underset{\smile}{\mathrm{sn}}\,(\overset{\frown}{u},k) = \frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{u}}\underset{\smile}{\sin}\,\overset{\frown}{\phi} = \left(\frac{\mathrm{d}}{\mathrm{d}\overset{\frown}{\phi}}\underset{\smile}{\sin}\,\overset{\frown}{\phi}\right)\frac{\mathrm{d}\overset{\frown}{\phi}}{\mathrm{d}\overset{\frown}{u}} \quad = \underset{\smile}{1}\cdot\underset{\smile}{\cos}\,\overset{\frown}{\phi}\sqrt{1 - k^2\underset{\smile}{\sin}^2\overset{\frown}{\phi}} = \underset{\smile}{1}\cdot\underset{\smile}{\mathrm{cn}}\,(\overset{\frown}{u},k)\,\underset{\smile}{\mathrm{dn}}\,(\overset{\frown}{u},k)$$

Conclusions

In pure mathematics, everything is just a number. There are no physical dimensions, not even for "angles." So even if we talk about a "radian" as a "unit," it's still just the pure dimensionless number 1.

On the other hand, in physics, angles ought to be a true dimension like any other. If we want to take advantage of dimensional analysis techniques, we need to be able to talk about "radians" and other angle units in a rigorous way that means something more than just a math abstraction.

For better or worse, SI has sided with the pure mathematicians. It has abdicated any notion of angle as a true dimension. This has lead to confusion and hand-waving as to what angular "units" should go into equations, and where. It has also lead to errors in physics computations because of confusion between quantities such as dimensionless-radians-per-second versus dimensionless-cycles-per-second (i.e., Hertz), or even versus dimensionless-counts-per-second.

Mohr et al (as well as other authors) have recognized this problem, and have tried to remedy it. To add clarity to this effort, I've offered terminology and notation that strongly marks the difference between pure numbers and "true" physical angles. I've also provided notation that strongly marks where we have "complete" versions of trigonmetric and other functions that operate on angles with true dimensionality, versus the "pure math" versions that work only on pure numbers.

When we use this notational style, and then apply calculus techniques to many physics problems that involve angles and trigonometry, we get rigorous resolutions to many dimensional-analysis quandaries involving angles. We don't need to resort to SI's abdication + hand-waving strategy.

In the process, we find time and time again that the constant that pops out to balance the dimensional analysis (after removing any window-dressing) turns out to be the (true) radian, and not the (true) turn. This reinforces the assertion that the (true) radian is naturally the coherent unit for angles as a physical dimension.