The Dozenal Society of America

A Dozen Properties of the Number Twelve

by Prof. Jay Schiffman

Introduction

The Ancient Greeks viewed the significance of the role of number throughout their rich scientific tradition. These peoples established a one-to-one correspondence between all entities in the universe and the notion of number. Briefly, all numbers are things and all things are numbers. It is of interest in mathematics today to link various disciplines with specific cardinal numbers. The following note presents an enumeration of a dozen properties of the integer one dozen in the branches of algebra and number theory. This list is far from exhaustive and the reader is invited to augment it. The basic definitions not presented in this list can readily be located in the papers referred to by references [3] and [5] in the appended bibliography.

Prof. Schiffman at the blackboard during the Dozenal Society of America's 1999 Annual Meeting at Nassau Community College, Long Island, NY.

Property 1;

One dozen is the **initial abundant number** [A, B]. An abundant number is a number which is smaller than the sum of its aliquot (proper) divisors. We note:

$$10; < (1 + 2 + 3 + 4 + 6) = 14;$$

[The first abundant numbers are:

{10;, 16;, 18;, 20;, 26;, 30;, 34;, 36;, 40;, 46;, 48;, 50;, … } [B]]

Property 2;

The dozen is **hypercomposite** [C, D, E]. An integer is termed hypercomposite if it possesses a greater number of divisors than any of its predecessors. One dozen has six divisors {1, 2, 3, 4, 6, 10;}, while each integer smaller than one dozen contains no more than 4 divisors.

[The first dozen highly composite numbers are:

{1, 2, 4, 6, 10;, 20;, 30;, 40;, 50;, X0;, 130;, 260;, … } [E]]

Property 3

The dozen represents the first number which is **neither a Converse Lagrange Theorem group** (CLT) **nor supersolvable** [F]. To explain these ideas in more lucid terms, let G be a finite group of order n. If for every integer d dividing n, G has a subgroup of order d, then G is called a CLT group. Moreover, if n is a positive integer such that every group of order n is CLT, then n is called a CLT number. A *supersolvable* number is defined analogously.

Property 4

The dozen represents the **minimal number of divisors for the cardinality of a nonabelian simple group** as well as a **nonsolvable group** and a **nontrivial perfect group**. A group G is styled perfect if $G' = G$ where G' is the subgroup generated by the set of commutators of the form $a^{-1}b^{-1}ab$ where $a, b \in G$.

Property 5

One dozen is the first group order possessing a greater number of (distinct) nonabelian isomorphism classes than abelian isomorphism classes. There exist three nonabelian isomorphism classes (corresponding to the alternating group A_4, the dihedral group D_6, and the dycyclic group) and two abelian isomorphism classes (corresponding to the cyclic group C_{10}, and the group $C_2 \oplus C_2 \oplus C_3$, where the symbol $\oplus$ denotes the group theoretic operation of direct sum). All of the above groups can be defined in terms of generators and defining relations encompassing the calculus of presentations [1, 8].

Two groups G and G^* are designated isomorphic if there exists a one-to-one correspondence between them preserving the group elements. More specifically, we are given a function f which establishes a one-to-one correspondence such that $f(a \,'\, b) = f(a) \,'\, f(b)$ for every $a, b \in G$. All groups which are isomorphic (structurally the same from an algebraic standpoint) belong to identical classes known as isomorphism classes.

Property 6

One dozen is the first natural number having a **perfect number of divisors** (six). A *perfect number* is a number enjoying the unique distinction that it coincides with the sum of its aliquot divisors [G]. Six is a perfect number; for 6 = (1 + 2 + 3). The origin of this concept dates back to the Ancient Greek Civilization. The universe was created by God, a perfect supreme being, in six days and on the seventh day, He rested. Thus the concept of "perfect number" evolved from theological pursuits.

Property 7

The cardinality of any nonabelian simple group must necessarily be divisible by at least one of the integers 10;, 14;, or 48;. This arithmetic fact was proven by W. Burnside in 111Ɛ; (1895.) (and is known decimally as the 12–16–56 Theorem) [2]. The dozen once again plays a premier role as the initial integer in the theorem.

Property 8

Consider the nonabelian simple alternating group of degree five, A_5. One dozen represents the largest cardinality for which a proper subgroup of A_5 exists. This idea is often stated as follows: A_5 possesses no subgroups of index 2, 3 or 4.

Property 9

The group SL(2, 3) of 2-square unimodular matrices (matrices of determinant one) over a field of cardinality three possesses no subgroup of cardinal number one dozen, the second smallest counterexample to the nonvalidity of the direct converse of Lagrange's Theorem on finite groups. One can demonstrate that SL(2, 3) is a group of order two dozen.

Property X

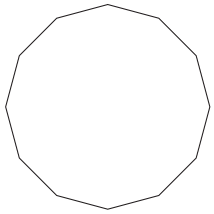
One dozen is the inital non square-free integer possessing a group (A_4) with trivial center. Such a group is called centerless.

Property Ɛ

If n is an odd integer multiple of one dozen, then n is not a Converse Lagrange Number in the sense that there exists at least one group of such order not satisfying the direct converse to Lagrange's Theorem on finite groups [3]. Moreover, if n is any integral multiple of one dozen, then n cannot be a supersolvable number. This follows since one can write the group order of one such group as $A_4 \times C_n$ (the direct product of A_4 with the cyclic group of order n, C_n) and note that $A_4 \times C_n$ contains the nonsupersolvable group A_4 as a subgroup. In lieu of the fact that subgroups of supersolvable groups are likewise supersolvable, the argument is complete.

Property 10

A regular dodecagon is constructible via a straightedge and compass alone. In fact, a regular n-gon is constructible in the above sense if the only odd primes dividing n are the Fermat primes whose squares do not divide n. A Fermat prime is a prime of the form $2^{2^n} + 1$, where n is a whole number [H]. The five presently known Fermat primes are {3, 5, 15, 195, 31Ɛ15} corresponding to $n = \{0, 1, 2, 3, 4\}$, respectively [J].



A regular dodecagon.

Reference [8] provides the theorem concerning when a regular n-gon is constructible using a straightedge and compass alone. See page 393. We note that the sole odd prime dividing 10; is 3, a Fermat prime, and $3^2 = 9$ does not divide 10;.

Bibliography

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Notes

A. Weisstein, Eric W. "Abundant Number", Wolfram Mathworld. Retrieved January 2011 <http://mathworld.wolfram.com/AbundantNumber.html>.

B. Sloane, N. J. A. "A005101 – OEIS", The On-Line Encyclopedia of Integer Sequences. The OEIS Foundation. Retrieved January 2011 <http://oeis.org/A005101>.

C. Weisstein's Mathworld definition of "highly composite number" is equivalent to Prof. Schiffman's "hypercomposite" (see Note D below): "Highly composite numbers are numbers such that divisor function $d(n) = \sigma_0(n)$ (i.e., the number of divisors of n) is greater than for any smaller n", where the divisor function is defined as "$\sigma_k(n)$ for n an integer is defined as the sum of the kth powers of the (integer) divisors of n".

D. Weisstein, Eric W. "Highly Composite Number", Wolfram Mathworld. Retrieved January 2011 <http://mathworld.wolfram.com/HighlyCompositeNumber.html>.

E. Sloane, N. J. A. "A002182 – OEIS", The On-Line Encyclopedia of Integer Sequences. The OEIS Foundation. Retrieved January 2011 <http://oeis.org/A002182>.

F. Weisstein, Eric W. "Lagrange's Group Theorem", Wolfram Mathworld. Retrieved January 2011 <http://mathworld.wolfram.com/LagrangesGroupTheorem.html>.

G. Weisstein, Eric W. "Perfect Number", Wolfram Mathworld. Retrieved January 2011 <http://mathworld.wolfram.com/PerfectNumber.html>.

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This document was remastered 25 January 2011 by Michael Thomas De Vlieger. The following were added: The photograph of Prof. Schiffman and caption, supplementary information to Properties 1 and 2, and the "Notes" section with online references. The word "twelve" in the text is replaced by either "the dozen" or "one dozen" to be consonant with Prof. Schiffman's use of dozenal notation throughout the text. The Fermat primes listed in Property 10; have been converted to dozenal rather than the decimal, to be consistent with Prof. Schiffman's prior uniform use of dozenal in the text.