On Maximal Repeating Sequence of Decimal Expansions in Base-Twelve

by T. J. Gaffney, DSA Member № 399; • 8 June 11Ɛ5;

Introduction

When learning the decimal expansions of fractions in the base-ten system, the first one that seems to be a huge pain is $1/7$. This decimal expansion is a worst case scenario, meaning that there could not possibly be fewer digits involved; we say that it has maximal repeating sequence. However, we'll learn that fractions of this type are actual somewhat friendly, in that any $m/7$, where m is an integer such that $0 < m < 7$, also has maximal repeating sequence with the digits being cyclical permutations of the $1/7$ case. In dozenal, $1/5$ and $1/7$ have this same maximal repeating sequence property and, for the same reason, have this same nice property.

Worst Case Scenario

I will demonstrate how $1/7$ is a worst case scenario, or "maximal repeating sequence", because it will come in useful later on.

When dividing one by any number you basically face an infinite number of zeros after the decimal. When doing long division you look at the first number, and after dividing the first zero by the divisor you take the remainder and "carry" it or move it front of the proceeding zero (which in decimal means to multiply by ten and add. For example:

```
     0.1
7 ) 1.000...
   - 7
     30
```

Where, in this case, on the bottom row 3 is the remainder and 0 is the carried down digit. This paper assumes that the reader knows how to do long division; I only provide the detail in case that the reader hadn't previously considered this 3 the remainder.

Now, there are very few values possible values for the remainder. Mathematicians will know by the Fundamental Theorem of Arithmetic and the layman will know by intuition, that this remainder cannot be less than zero and that it cannot be greater than or equal to the divisor (7 in the example). If the remainder is not less than the divisor, then you increment the solution you've listed (the 1 in the example) until it is.

If you ever get a zero as the remainder, then it is clear to see (when dividing along infinite zeros) that the remaining solutions will be zeros, which we don't write because they're after the decimal. This is called a terminating fraction. (Try the calculation of $1/8$ for an example.) The other option is that the remainder has already been encountered in the long division, in which case (since we're working along infinite zeros, that all act the same) it is clear that the solutions would simply repeat themselves. This is a repeating sequence. We see this in the $1/7$ case (here I write the remainder as super-scripts to the zeros to save space; the way you would see them in subtraction):

$$\begin{array}{r l} & 0\,.\,1\;\;4\;\;2\;\;8\;\;5\;\;7\;\;1\;\;\ldots \\ 7 & \overline{)\,1\,.{}^{1}0\;{}^{3}0\;{}^{2}0\;{}^{6}0\;{}^{4}0\;{}^{5}0\;{}^{1}0\;\ldots} \end{array}$$

Obviously, the pattern will continue forever.

Since, there is a finite option for remainders, then the solution (for fractions with one as the dividend) must repeat or terminate. (In fact, it is a well known mathematical fact that the decimal expansions of all fractions must repeat or terminate. The proof is a generalization of the preceding arguments.) Further an expansion of this type (if not terminating) must repeat within *m* digits after the decimal place, where m is the divisor. That is to say that at most *m*-1 digits will be repeated, since we don't allow zero to be a remainder (for that would be a termination), then there are only *m*-1 unique remainders. This produces the maximal repeating sequence (our "worst case scenario").

A Nice Property

It is a fun bit of trivia that every mathematician should know that multiples of $1/7$ use permutations of the same repeating sequence. That is to say that the repeated digits are the same, but in different order. We show the first six digits of the sevenths in decimal in Figure 1 at right.

$1/7 = 0.142857\ldots$
$2/7 = 0.285714\ldots$
$3/7 = 0.428571\ldots$
$4/7 = 0.571428\ldots$
$5/7 = 0.714285\ldots$
$6/7 = 0.857142\ldots$

Fig. 1: Decimal Sevenths

The reader may immediately see why this is. A hint for mathematicians is to notice that the permutations are of the particular, cyclic type.

The proof of this actually quite simple. In the first example, the calculation of $1/7$, I write the 1 in front of the first zero in the dividend, as though it is the remainder when dividing only the digit 1 by 7. (Although, it is usually taught to just start the division by looking at the 1 and 0 together for 10, what you're really doing here is just exactly the same "carrying" of the remainder that you're doing everywhere else.) But, if instead of a 1 for the first digit there was a 2 or 3 or some other number (as it would be in these other fractions), then the first remainder would also be this other number.

From there, and because you're moving along infinite zeros, the division would proceed exactly as it did in the $1/7$ case, with the same repeating sequence as well, but merely starting at a different place.

If it is not immediately clear why these expansions use the same 6 digits, it may be a useful exercise for the reader to calculate a few until he or she can see the pattern on his or her own.

Dozenal

$1/2 = 0;6$
$1/3 = 0;4$
$1/4 = 0;3$
$1/6 = 0;2$
$1/8 = 0;16$
$1/9 = 0;14$

Fig. 2: Some Dozenal Terminating Fractions

A driving argument for the usage of dozenal is the ease of expanding fractions. The fractions shown in Figure 2 at left have much cleaner and easier expansions in dozenal.

But this argument seems to be weakened by two fractions, $1/5$ and $1/7$. One seventh is just as "bad" in base-twelve as it is in base-ten, and one fifth seems much worse. However, by simply calculating these two dozenally, you can see that these both have the property of having a maximal repeating sequence, and thus inherit this nice property of just cyclically repeating the same digits.

The expansions of the fifths and sevenths in dozenal appear in Figures 3 and 4 below.

What's So Nice?

$1/5 = 0;2497\ldots$
$2/5 = 0;4972\ldots$
$3/5 = 0;7249\ldots$
$4/5 = 0;9724\ldots$

Fig. 3: Dozenal Fifths

$1/7 = 0;186\text{X}35\ldots$
$2/7 = 0;35186\text{X}\ldots$
$3/7 = 0;5186\text{X}3\ldots$
$4/7 = 0;6\text{X}3518\ldots$
$5/7 = 0;86\text{X}351\ldots$
$6/7 = 0;\text{X}35186\ldots$

Fig. 4: Dozenal Sevenths

$1/8 = 0.125$
$2/8 = 0.25$
$3/8 = 0.375$
$4/8 = 0.5$
$5/8 = 0.625$
$6/8 = 0.75$
$7/8 = 0.875$

Fig. 5: Decimal Eighths

"So, what's so great about the maximal repeating sequence property? I still say that $1/7$ is the hardest to memorize." Try to memorize your eighths in decimal ($1/8$ to $7/8$, shown in Figure 5 below) and you'll find that you'll be memorizing more numbers than when you memorize the sevenths.

When memorizing the sevenths in decimal (this will work the same with the fifths and sevenths in dozenal). You need to first memorize $1/7 = 0.142857$. It's not pretty, but it is only six digits. Now say you want to recall $m/7$ on cue. You only need to remember the first digit of this expansion, because, as we know, the rest cyclically follows the order in the $1/7$ case. However, this is easy; since the leading digits of the numbers $1/7$ to $6/7$ must be in increasing numerical order. (Because, otherwise, it would suggest that $2/7$ is less than $1/7$, for example.)

Say you want to impress your friends by "calculating" 6/7. You start by looking at the digits in $1/7$: 142857 in numerical order: $1 < 2 < 4 < 5 < 7 < 8$; since 8 is the 6th smallest here, you take that to be the leading digit. Then you recall the rest of $1/7$, in order: 0.857142 (Where, upon reaching the "end", or the 7 digit, you just continue from the beginning.)

For example, you might use this method to calculate the fifths in dozenal, where $1/5$: 0;2497. Which gives you $2 < 4 < 7 < 9$. Thus we obtain the leading digits of the fractions as seen at left below. Adding the remaining digits cyclically, we obtain the situation seen at right below:

$1/5 = 0;2\ldots$	$1/5 = 0;2497\ldots$
$2/5 = 0;4\ldots$	$2/5 = 0;4972\ldots$
$3/5 = 0;7\ldots$	$3/5 = 0;7249\ldots$
$4/5 = 0;9\ldots$	$4/5 = 0;9724\ldots$

Thus it is easy to recall these seemingly unfriendly fractions. For this reason, along with many others, dozenal is a superior counting system.

featured figures

1/7

Building on Mr. Gaffney's examination of dozenal fractions, we thought about other such fractions. Here we explore these for you, first examining the first one dozen ten reciprocals and their unique multiples. In the tables below, all reciprocal multiples ignore zero in the unit place or "integer part" of the figure. Each reciprocal multiple appears in one of three states. The first state terminates, like ⅔ = ;8. The second state repeats after the unit point, like ⅖ = ;4972.. , the repetition indicated by ellipsis (..). The last is a repeating series after an initial quantity of digits, like 5/12 = 0;4⁝35186X.., the repeating series appearing here between a vertical series of dots (⁝) and the ellipsis. The reciprocals of Ɛ and 11; are fun. Do you recognize their patterns? One dozen five displays the behavior Mr. Gaffney described for fifths and one sevenths. Can you explain some of the pattern changes that happen, say, in the multiples of the reciprocal of 17; (19.)?

Halves

2	½	;6

Thirds

3	⅓	;4
	⅔	;8

Quarters

4	¼	;3
	¾	;9

Fifths

5	⅕	;2497..
	⅖	;4972..
	⅗	;7924..
	⅘	;9247..

Sixths

6	⅙	;2
	⅚	;X

Sevenths

7	1/7	;186X35..
	2/7	;35186X..
	3/7	;5186X3..
	4/7	;6X3518..
	5/7	;86X351..
	6/7	;X35186..

Eighths

8	⅛	;16
	⅜	;46
	⅝	;76
	⅞	;X6

Ninths

9	1/9	;14
	2/9	;28
	4/9	;54
	5/9	;68
	7/9	;94
	8/9	;X8

Tenths

X	1/X	;1⁝2497..
	3/X	;3⁝7249..
	7/X	;8⁝4972..
	9/X	;X⁝9724..

Elevenths

Ɛ	1/Ɛ	;1..	6/Ɛ	;6..
	2/Ɛ	;2..	7/Ɛ	;7..
	3/Ɛ	;3..	8/Ɛ	;8..
	4/Ɛ	;4..	9/Ɛ	;9..
	5/Ɛ	;5..	X/Ɛ	;X..

10;	1	;1	7	;7
	5	;5	Ɛ	;Ɛ

11;	1	;0Ɛ..	5	;47..	9	;83..
	2	;1X..	6	;56..	X	;92..
	3	;29..	7	;65..	Ɛ	;X1..
	4	;38..	8	;74..	10	;Ɛ0..

12;	1	;0⁝X35186..
	3	;2⁝6X3518..
	5	;4⁝35186X..
	9	;7⁝86X351..
	Ɛ	;9⁝5186X3..
	11	;Ɛ⁝186X35..

13;	1	;0⁝9724..	8	;6⁝4972..
	2	;1⁝7249..	Ɛ	;8⁝9724..
	4	;3⁝2497..	11	;X⁝4972..
	7	;5⁝7249..	12	;Ɛ⁝2497..

14;	1	;09	9	;69
	3	;23	Ɛ	;83
	5	;39	11	;99
	7	;53	13	;Ɛ3

15;	1	;085792 14Ɛ364 29X7..
	2	;14Ɛ364 29X708 5792..
	3	;214Ɛ36 429X70 8579..
	4	;29X708 579214 Ɛ364..
	5	;36429X 708579 214Ɛ..
	6	;429X70 857921 4Ɛ36..
	7	;4Ɛ3642 9X7085 7921..
	8	;579214 Ɛ36429 X708..
	9	;6429X7 085792 14Ɛ3..
	X	;708579 214Ɛ36 429X..
	Ɛ	;79214Ɛ 36429X 7085..
	10	;857921 4Ɛ3642 9X70..
	11	;9214Ɛ3 6429X7 0857..
	12	;9X7085 79214Ɛ 3642..
	13	;X70857 9214Ɛ3 6429..
	14	;Ɛ36429 X70857 9214..

16;	1	;08	Ɛ	;74
	5	;34	11	;88
	7	;48	15	;Ɛ4

17;	1	;076Ɛ45..	X	;639582..
	2	;131X8X..	Ɛ	;6Ɛ4507..
	3	;1X8X13..	10	;76Ɛ450..
	4	;263958..	11	;826395..
	5	;31X8X1..	12	;8X131X..
	6	;395826..	13	;958263..
	7	;45076Ɛ..	14	;X131X8..
	8	;5076Ɛ4..	15	;X8X131..
	9	;582639..	16	;Ɛ45076..

18;	1	;0⁝7249..	Ɛ	;6⁝7249..
	3	;1⁝9724..	11	;7⁝9724..
	7	;4⁝2497..	15	;X⁝2427..
	9	;5⁝4972..	17	;Ɛ⁝4972..

19;	1	;0⁝6X3518..	Ɛ	;6⁝35186X..
	2	;1⁝186X35..	11	;7⁝5186X3..
	4	;2⁝35186X..	14	;9⁝186X35..
	5	;2⁝X35186..	15	;9⁝86X351..
	8	;4⁝6X3518..	17	;X⁝X35186..
	X	;5⁝86X351..	18	;Ɛ⁝5186X3..

1X;	1	;0⁝6..	11	;7⁝1..
	3	;1⁝7..	13	;8⁝2..
	5	;2⁝8..	15	;9⁝3..
	7	;3⁝9..	17	;X⁝4..
	9	;4⁝X..	19	;Ɛ⁝5..