SOME NOTES ON THE HISTORY AND DESIRABILITY OF USING ALTERNATE NUMBER BASES IN ARITHMETIC

by Christopher J. Osburn

We all do arithmetic. We do it in the supermarket when providing for our families. We do it on the highway when comparing our speed with the posted limits (sometimes). We do it in the restaurant when determining how much of a tip to leave on the table, or whether we'll have to wash dishes to pay for the meal. Arithmetic, and mathematics as a whole, is always around us from the most mundane tasks to the most embarrassing and profound situations.

With some minor exceptions we perform arithmetic operations in base ten. But is base ten really the best way to do arithmetic? Are calculations easier to perform in some other base, say twelve or sixteen? Let us take a brief look at some of the inherent advantages and disadvantages of the use of alternate number bases in arithmetic, starting with our tried and true friend, base ten.

The primary advantage that base ten gives us is that we're accustomed to it. The most popular explanation is that we have ten fingers on our hands. We are able to match our fingers to up to ten of some other object. Counting the number of times we can do this before we run out of whatever we were counting allows us to use numbers greater than ten. Some cultures, for similar reasons, have used number systems of five and twenty (the latter by calling the toes into play).[1] There are some disadvantages that are immediately apparent in these bases. Base five, for instance, is a fairly small base, which leads to long strings of numerals even for small values. (Compare one hand and three fingers (13_{five}) to eight fingers (8_{ten}) Base twenty has the problem that very few of us can bend our toes independently of the others. Peoples in colder climes may have to remove shoes of moccasins to do any such counting. Whatever the reason, our familiar decimal system predominates.

There are some notable exceptions. Ancient Sumerians and Babylonians used a sexagesimal (base sixty) system of enumeration in connection with the place-value system. Each sexagesimal place, however, was constructed of cuneiform symbols giving the number of tens and units for that place.[2] Some Northern European societies had a quantity known as a "great hundred" made up of ten dozens (decimal 120.), reflecting the rudiments of a duodecimal (base twelve) counting system.[3] The Romans, even though they used base ten for their integer counting, had a system of duodecimal fractions. It is believed they chose this because of easy divisibility in so many different ways.[4]

Despite our use of the decimal system for many millennia, there is something that requires us to consider non-decimal enumerating: the electronic digital computer.

Computers, at their lowest levels of operation, know only whether a current is flowing through a transistor or not. This off/on choice leads us to the binary (base two) system of numeration. But while computers have little problem working in binary, for humans it can be a bit cumbersome. For example:

$$842_{ten} = 1101001010_{two}$$

As you can see, relatively small numbers in decimal produce some real monster-sized

binary numbers. To cope, we have developed some convenient shortcuts. By converting binary numbers into octal (base eight) or hexadecimal (base sixteen) numbers, we make binary numbers more manageable for humans. This is actually quite easy. Taking our example from above,

$$1101001010_{two}$$

we divide the number up three places at a time from the right,

$$1\ 101\ 001\ 010$$

and then convert each group of three into single octal digits by finding the values that correspond to each place:

1	101	001	010
1	5	1	2

This gives us:

$$842_{ten} = 1512_{eight}$$

The process for converting binary into hexadecimal is similar; start by dividing the number into four digit groups:

$$11\ 0100\ 1010$$

and insert the appropriate values:

11	0100	1010
3	4	10

This leads to a bit of a problem. How do we squeeze that ten into a single digit? The current usage in the computer industry is to represent the values ten through fifteen by the letters "A" through "F": "A" equals ten, "B" equals eleven, etc. Our conversion from above then becomes:

11	0100	1010
3	4	A

giving us:

$$842_{ten} = 34A_{sixteen}$$

In some bases, identifying prime numbers *greater than 2* and perfect squares (or at least ruling them out) is fairly easy, in others it is more difficult. A good test is to check the final digit in the number. For example, in base ten we know there are no prime numbers ending with the numeral 4 and there are no perfect squares that end with a 7. How many of the available numerals in a given base can terminate a prime number? How many will terminate a perfect square? It is also useful to compare that number with the total available. If, for instance, a prime number can end with any digit at all, that test becomes useless.

We have so far come across several different numbering systems, which we can categorize as follows:

1. The "Finger" Bases: five, ten, twenty;
2. The Binary Bases: two, eight, sixteen;
3. Other Bases: twelve and sixty.

Table 1: Rules of Divisibility for Selected Bases

Base 2:
- 2: A number is even if it ends in 0, odd if it ends in 1

Base 5:
- 2: Any number whose digits add to a multiple of 2
- 4: Any number whose digits add to a multiple of 4

Base 8:
- 2: Any number ending in an even digit
- 4: Any number ending in 0 or 4
- 7: Any number whose digits add to a multiple of 7

Base 10:
- 2: Any number ending in an even digit
- 3: Any number whose digits add to a multiple of 3
- 5: Any number ending in 0 or 5
- 6: Any even number whose digits add to a multiple of 3
- 9: Any number whose digits add to a multiple of 9

Base 12:
- 2: Any number ending in an even units place
- 3: Any number ending in 0, 3, 6, 9
- 4: Any number ending in 0, 4, 8
- 6: Any number ending in 0, 6
- 11: Any number whose digits add to a multiple of 11

Base 16:
- 2: Any number ending in an even units place
- 3: Any number whose digits add to a multiple of 3
- 4: Any number ending in 0, 4, 8, 12
- 5: Any number whose digits add to a multiple of 5
- 6: Any even number whose digits add to a multiple of 6
- 8: Any number ending in 0 or 8
- 10: Any even number whose digits add to a multiple of 5
- 15: Any number whose digits add to a multiple of 15

Base 20:
- 2: Any number ending in an even units place
- 4: Any number ending in 0, 4, 8, 12, 16
- 5: Any number ending in 0, 5, 10, 15
- 10: Any number ending in 0, 10
- 19: Any number whose digits add to a multiple of 19

Base 60:
- 2: Any number ending in an even units place
- 3: Any number whose units place is a multiple of 3
- 4: Any number whose units place is a multiple of 4
- 5: Any number whose units place is a multiple of 5
- 6: Any number whose units place is a multiple of 6
- 10: Any number ending in 0, 10, 20, 30, 40, 50
- 12: Any number ending in 0, 12, 24, 36, 48
- 15: Any number ending in 0, 15, 30, 45
- 20: Any number ending in 0, 20, 40
- 30: Any number ending in 0, 30
- 59: Any number whose digits add to a multiple of 59

Base	Prime Number End Digits	%	Perfect Square End Digits	%
2	1	100	0, 1	100
5	1, 2, 3, 4	200	0, 1, 4	60
8	1, 3, 5, 7	100	0, 1, 4	38
10	1, 3, 7, 9	80	0, 1, 4, 5, 6, 9	60
12	1, 5, 7, 11	67	0, 1, 4, 9	33
16	1, 3, 5, 7, 9, 11, 13, 15	100	0, 1, 4, 9	25
20	1, 3, 7, 9, 11, 13, 17, 19	80	0, 1, 4, 5, 9, 16	30
60	1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59	53	0, 1, 4, 9, 16, 21, 24, 25, 36, 40, 45, 49	20

Table 2: Prime Number End Digits (Totatives) and Perfect Square End Digits.

We have also seen that some bases are good for humans while others are good only for computers. Is there some way we can quantify the usefulness of these systems (for humans) so that we can compare them? Which of these bases is really the best for counting and arithmetic for humans?

One way of comparing number bases is to compare some of their divisibility indicators. For example, a divisibility indicator in base ten would be the fact that all numbers divisible by five end in a zero or a five digit. Easy rules like this are one way we make counting and arithmetic easy on ourselves. George Terry, in his book *Duodecimal Arithmetic*, suggests tests to help identify prime numbers and perfect squares.[5]

Divisibility Rules.

Let us take a quick look at the divisibility rules first. We will concentrate on the "easy" rules (hard rules aren't that valuable to humans). We'll restrict ourselves to numbers less than the base number itself (except for base two). Table 1 on page 19; shows when a number in the given base is divisible by the digit in the left hand column.

End Digits of Prime and Square Numbers.

Note: the columns marked "%" in Table 2 refer to the percentage of digits that appear against the given base. The percentages given after the prime digit column refer to the number of odd digits that appear. Base 5 reads 200% in this column, as numbers ending in even digits can also be prime.

Regularity of Digits.

Table 3 on page 1Ɛ; lists "regular numbers" along with a "regularity index" for each base. A regular number is a number, in base sixty, the reciprocal of which has a finite number of places. We can extend this concept to any other base and say a regular number has a terminating fractional part in that base. For example, ⅓ is a terminating fraction in base twelve (0.4_{twelve}) but it is not a terminating fraction in decimal ($0.333\ldots_{\text{ten}}$). So, three is a regular number in base twelve but not in base ten. This is a good alternative to counting the divisibility rules presented in Table 1. If we look at every single-digit number greater then 1 in each base we can see what portion of them are regular. We call that portion the "regularity index" and express it as a percentage.

Base	Regular Numbers	Regularity Index (%)
2	[none]	0
5	[none]	0
8	2, 4	33
10	2, 4, 5, 8	50
12	2, 3, 4, 6, 8, 9	60
16	2, 4, 8	21
20	2, 4, 5, 8, 10, 16	33
60	2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 27, 30, 32, 36, 40, 45, 48, 50, 54	41

Table 3: Regular Numbers for Selected Bases. *Editor's Note: this table includes regular numbers by the Author's definition which are less than the base given in the leftmost column. Such positive integers lesser than the base would thus be single digits in that base.*

Comparing the Bases.

We have quite a lot of data to digest. Let's look at how we might combine our indices and percentages into something we can use for comparisons. This will be somewhat subjective since we're really trying to quantify how a human will feel about each number base while counting and doing arithmetic.

We should give a positive consideration to the regularity index, since we'd like to avoid infinite fractions. We'll give a smaller positive consideration to the fact that a larger base yields a more compact notation; the length of a numeral gets shorter or remains unchanged as the log of its base increases. (We use the natural logarithm to avoid showing preference to any integer base.)

Negative consideration should be given for the number of different digits that are found at the ends of prime and square numbers (fewer is better). And we'll consider the size of the multiplication table. A bigger base has a larger table to learn and we should think of the school kids. Combining all these influences gives us the following relation:

$$I = \frac{R \times \ln b}{P \times S \times b}$$

where:

b is the base in question

R is the regularity index

P is the percentage of odd digits found at the ends of prime numbers

S is the percentage of all digits found at the ends of perfect squares

and the percentages P, R and S are expressed as fractionals.

Base b	Index I
2	.000
5	.000
8	.231
10	.240
12	.559
16	.149
20	.208
60	.265

Table 4: Compare the bases.

This yields the data shown in Table 4.

This table indicates that base twelve is, by far, a much more logical base to do arithmetic in. Bases eight, ten and sixty seem to have fared about equally well (even with base sixty's enormous multiplication table). It would seem that base twelve ranked so much higher because it combines good divisibility patterns (noted by the regularity index) with a fairly small set of operation tables.

On the other hand, note bases two and five bringing up the rear. For base five, there are no terminating decimal fractions. Also, as an odd-numbered base, we have more difficulty finding odd and even numbers in base five. A base five prime number may end in any digit. For example: 31_{five} has an odd last digit, but is equal to 16_{ten}, an even number. Base two fails mainly because it is so cumbersome to work with, and that it's more difficult to guess whether a number might be prime or square (shown by high values of P and S). The regularity index of base two, zero, may be merely a problem in defining the regularity index. There are simply no integers between 1 and 1. Arbitrarily setting the regularity index to 50% gives a final index value of .173. This is still quite low, but seems more appropriate.

Of the bases we haven't considered, does anything else compare to base 12? Base 6 does with an index of .504. These are the only two bases that come in above .500 and, in fact, the only two coming in above .400. (Base 4 came in third at .347.) The top ten are:

12, 6, 4, 24, 30, 18, 60, 10, 36, 8.

Should we convert to base twelve? Re-educating several billion people seems like a daunting task, so we might begin by teaching duodecimals in parallel with decimal math to children just entering school. In my fifth year at elementary school, I volunteered to teach octal arithmetic to the class. My classmates reacted positively, having fun playing with slightly altered arithmetic rules and viewing the world through the eyes of an eight-fingered creature. Today I carry out counting tasks in parallel with decimal and dozenal, which provides me a reality check of sorts. Giving people another lens thorough which to see the world will do no harm and may well be of great benefit.

Notes

[1] Eves, Howard; *An Introduction to the History of Mathematics*, fifth edition, Philadelphia: Saunders College, 1982; p. 4

[2] Ibid, p. 10

[3] Menninger, Karl; *Number Words and Number Symbols*, English translation, Cambridge, Massachusetts: The MIT Press, 1969; pp. 154ff.

[4] Ibid, pp. 158ff.

[5] Terry, George S.; *Duodecimal Arithmetic*, London: Longmans Green and Co., 1938

Editor's note: The text of this article can be retrieved at http://www.ubergeek.org/~chris/random/base12.html