The Dozenal Society of America

Music, Scales, and Dozens

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Dr. Impagliazzo tickles the keys at the Dozenal Society of America's 1985 Annual Meeting at Nassau Community College, Long Island, NY.

Part I.
Mathematical Considerations and Pythagorean Scales

<u>Introduction</u>. In Plato's *Republic*, Socrates expresses the following to Glaucon concerning music:

The beauty of style and harmony and grace and good rhythm depends on simplicity. I mean the true simplicity of a rightly and nobly ordered mind and character, not that other simplicity which is only a euphemism for folly. [Jowett, p. 104].

Surely, in contrasting the works of early mathematicians such as Euclid and Pythagoras with early composers such as Antonio Vivaldi and J. S. Bach, one cannot but pause and reflect on the beauty and simplicity of perfection of the work of these persons of genius.

<u>Mathematical Considerations</u>. When a string is set into vibration, it produces overtones or harmonics. These overtones occur as multiples of their corresponding frequencies. For example, the first overtone of a fundamental frequency or **tonic** is its **octave**, which is twice the fundamental frequency. The second overtone is three times the frequency of the tonic. Interestingly enough, these ratios form a sequence:

$$1/1,\ 2/1,\ 3/1,\ 4/1,\ 5/1,\ 6/1,\ 7/1 \ldots$$

which is an **arithmetic progression**. Their reciprocals

$$1/1,\ 1/2,\ 1/3,\ 1/4,\ 1/5,\ 1/6,\ 1/7 \ldots$$

form a **harmonic progression**. The frequency of overtones can be formed by the proportion

$$f_{(n+1)} / f_n = (n+1) / n$$

where f_n and $f_{(n+1)}$ are the frequencies of the n^{th} and the $(n+1)^{th}$ overtones respectively [Lawlis, p. 593]. Thus, using a tonic frequency f_1 = 110 Hertz,

$f_1 = 1/1 \times (110) = 110$	$f_5 = 5/1 \times (110) = 550$
$f_2 = 2/1 \times (110) = 220$	$f_6 = 6/1 \times (110) = 660$
$f_3 = 3/1 \times (110) = 330$	$f_7 = 7/1 \times (110) = 770$
$f_4 = 4/1 \times (110) = 440$	$f_8 = 8/1 \times (110) = 880$, etc.

All octave frequencies are a power of two of the tonic. Specifically, these are the f_2, f_4, f_8, etc. The ratios formed by these frequencies are:

$f_1/f_1 = (110)/(110) = 1/1$	$f_5/f_4 = (550)/(440) = 5/4$
$f_2/f_1 = (220)/(110) = 2/1$	$f_6/f_4 = (660)/(440) = 3/2$
$f_3/f_2 = (330)/(220) = 3/2$	$f_7/f_4 = (770)/(440) = 7/4$
$f_4/f_4 = (440)/(440) = 1/1$	$f_8/f_8 = (880)/(880) = 1/1$

$$f_9/f_8 = (990)/(880) = 9/8$$

Clearly, relative to octaves, the ratios are those of small numbers.

<u>Pythagorean Scales</u>. Overtones of a vibrating string produce tones that are harmonious with the tonic. The octaves are the most harmonious. Other than octaves, the tone of the best consonance with the tonic is that which is in ratio of 3/2 of the tonic and the octaves of 3/2 of the tonic as shown in Figure 1 [Pierce, p. 65]. Pythagoras reasoned that since the ratio 3/2 produced the tone of best consonance, then 3/2 of 3/2 or 9/4 is the next tone of best consonance. Keeping tones within a single octave, divide 9/4 by 2. Hence, this proper tone has a relative value of 9/8.

The frequencies of two notes (such as c and g) that are an interval of a **fifth** apart are always in the ratio of 100:150. Their harmonics will be in the same ratio; so the second, fourth, and so forth, harmonics of c will coincide with the third, sixth, and so forth, harmonics of g, because at each of these coincidences, $3f_0 = 2(3/2)f_0$.

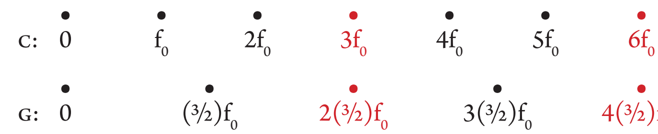


Figure 1.

In a similar fashion, 3/2 of 9/8 is 27/16 is a ratio of the next proper tone. Continuing, 3/2 of 27/16 is 81/32, is a next proper tone. Since 81/32 (2 17/32) is greater than 2 (the octave), divide this ratio by 2 so that the tone of ratio 81/64 is a next proper tone. Similarly, 3/2 of 81/64 or 243/128 is a next proper tone.

The difference between an octave and a fifth (ratio = 3/2) is called a **fourth**. Therefore, instead of multiplying the tonic 1 by 3/2, divide the octave by 3/2. Hence, (2/1)/(3/2) = 4/3. Based on the tonic, the ratios of best consonance with the tonic are 2/1, 3/2, and 4/3 which are respectively the octave, the fifth, and the fourth [Malcolm, p. 612].

Placing the ratios of best consonance and of proper tone in mathematical order results in:

$$1/1 < 9/8 < 81/64 < 4/3 < 3/2 < 27/16 < 243/128 < 2/1$$

Tones corresponding to these ratios form the Diatonic scale of Pythagoras. Using the familiar {Do, Re, Mi, Fa, Sol, La, Ti, Do} to identify the corresponding ratios and using the note c as the tonic produces the diatonic scale as shown in Figure 2.

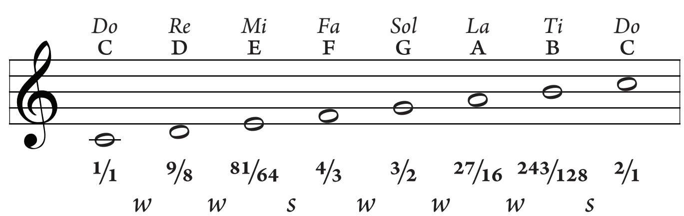


Figure 2.

Part II.
Just Intonation and the Chromatic (Dozenal) Scale

<u>The Scale of Just Notation</u>. Considering the harmonic of a vibrating string, the octave (first harmonic), the fifth (third harmonic), and the third (fifth harmonic) are most prominent while at the same time, the most concordant. These have frequency ratios of 1/1 for the tonic, 2/1 for the octave, 3/2 for the fifth, and 5/4 for the third. Hence, the most natural concordant harmony would be a combination of these three tones. Thus, a tonic chord as c–e–g–c would have the frequency ratios of {1/1, 5/4, 3/2, 2/1}, respectively.

If the same scheme is started with the fifth, its ratios become {(1/1)(3/2), (5/4)(3/2), (3/2)(3/2), (2/1)(3/2)} or {3/2, 15/8, 9/4, 3/1}. This represents the dominant chord g–b–d–g. If the scheme is started with the fourth, which is the inverted fifth with ratio $1/(3/2) = 2/3$, the ratios are $\{(1/1)(2/3), (5/4)(2/3), (3/2)(2/3), (2/1)(2/3)\}$ or $\{2/3, 5/6, 1/1, 4/3\}$. This represents the subdominant chord F–A–C–F. Summarizing the previous discussion, the three sequences of tones are:

Thus, a tonic chord as C–E–G–C would have the frequency ratios of $\{1/1, 5/4, 3/2, 2/1\}$, respectively.

$$\begin{aligned} \text{C–E–G–C} &= 1/1 \times \{1/1, 5/4, 3/2, 2/1\} = \{1/1, 5/4, 3/2, 2/1\} \\ \text{G–B–D–G} &= 3/2 \times \{1/1, 5/4, 3/2, 2/1\} = \{3/2, 15/8, 9/4, 3/1\} \\ \text{F–A–C–F} &= 2/3 \times \{1/1, 5/4, 3/2, 2/1\} = \{2/3, 5/6, 1/1, 4/3\} \end{aligned}$$

When these ratios are adjusted so that they fall within the range of one octave and in ascending order, they form the **Scale of Just Intonation** [Coxeter, p. 318]. This is shown in Figure 3.

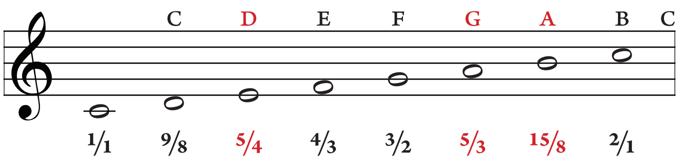


Figure 3.

James Jeans wrote:

> **It is found to be a quite general law that two tones sound well together when the ratio of their frequencies can be expressed by the use of small numbers, and the smaller the numbers the better is the consonance.** [Jeans, p. 154].

The Scale of Just Intonation is considered "purer" than the Pythagorean Scale because it is composed of ratios of numbers which are composed of smaller numbers.

<u>Defects in Tonal Scales</u>. In the Pythagorean Scale, the ratio between successive notes is $9/8 = 1.12500$ (e.g. $243/128 \div 27/16$) while the ratio between successive semitones is $256/243 = 1.05350$ (e.g. $4/3 \div 81/64$). [Jeans, p. 167]. Two successive semitones do not equal a whole tone. This is a cause for musical distortion especially when modulating from one tonic key to another.

In the Scale of Just Intonation, semitone ratios such as F to E and C to B have constant ratios equal to 16/15. However, the whole tone ratios are not constant. For example, the ratio E to D results in the ratio 5/4 to 9/8 which equals 10/9. The ratio B to A, however, has the ratio 15/8 to 5/3, which equals 9/8. Clearly, $9/8 \neq 10/9$. Music played in a key which is tuned to Just Intonation would sound optimal. A change in key, however, would create a definite tonal distortion.

<u>The Chromatic Scale, or Scales and Dozens</u>. Suppose successive ratios of 3/2 are taken from a fundamental tonic of ratio 1. Then new notes are generated (those with sharps) to form what is called the **Chromatic Scale**, which contains one dozen notes. If the tonic is the note C, then:

$(3/2)^{-1}$ corresponds to **F**	$(3/2)^{6}$ corresponds to **F#**
$(3/2)^{0}$ corresponds to **C**	$(3/2)^{7}$ corresponds to **C#**
$(3/2)^{1}$ corresponds to **G**	$(3/2)^{8}$ corresponds to **G#**
$(3/2)^{2}$ corresponds to **D**	$(3/2)^{9}$ corresponds to **D#**
$(3/2)^{3}$ corresponds to **A**	$(3/2)^{10}$ corresponds to **A#**
$(3/2)^{4}$ corresponds to **E**	$(3/2)^{11}$ corresponds to **E# = F**
$(3/2)^{5}$ corresponds to **B**	$(3/2)^{12}$ corresponds to **B# = C**

These values, when normalized within a single octave produce what is commonly known in music theory as a **circle of fifths**. However, the value $(3/2)^{12} = 129.74634$ while the value $(2/1)^7 = 128$. Herein lies the discrepancy. A dozen factors of 3/2 should have encompassed seven octaves. It did not. Over seven octaves there exists a ration of $129.74634/128 = 1.01364$ (often called the "comma of Pythagoras") rather than the ideal 1.

In order to compromise the aforementioned discrepancies, a scale based on a dozen semitones each possessing a ratio of $2^{(1/12)} = 1.05946$ was constructed. In this case, the ratio of any two adjacent notes on the scale would equal a constant. As a case in point, the construction of a fifth from a given tonic would be $2^{(7/12)} = 1.49831$, very close but less than the ideal of $3/2 = 1.50000$. The third, which is four semitones from the tonic, is $2(4/12) = 1.25992$. This differs from both the Pythagorean third of $81/64 = 1.26563$ and the Just Intonation third of $5/4 = 1.25000$.

The scale based on a dozen semitones of ratio $2^{(1/12)}$ is called the **Well-Tempered Scale**. A table of ratios of the dozen notes is shown:

C	:	2^{0}	= 1.00000	**G**	:	$2^{(7/12)}$	= 1.49831
C#	:	$2^{(1/12)}$	= 1.05946	**G#**	:	$2^{(8/12)}$	= 1.58740
D	:	$2^{(2/12)}$	= 1.12246	**A**	:	$2^{(9/12)}$	= 1.68179
D#	:	$2^{(3/12)}$	= 1.18921	**A#**	:	$2^{(10/12)}$	= 1.78180
E	:	$2^{(4/12)}$	= 1.25992	**B**	:	$2^{(11/12)}$	= 1.88775
F	:	$2^{(5/12)}$	= 1.33484	**C**	:	$2^{(12/12)}$	= 2.00000
F#	:	$2^{(6/12)}$	= 1.41421				

Note that with the Well-Tempered, dozenal scale the tones C#, D#, F#, G#, and A# could be replaced by D♭, E♭, G♭, A♭, and B♭, respectively. In the Pythagorean or the Scale of Just Intonation this is impossible.

The Well-Tempered scale has its defects also. Its greatest defect is that it is not always harmonious. It is always "almost in tune". A vocalist or instrumentalist such as a violinist will sing or play toward notes of Just Intonation. A piano is almost always tuned to Well-Tempered balance. Singing or playing an instrument together with a piano sometimes presents problems, since it will always sound a "little" out of tune. The greatest asset of the Well-Tempered Scale is that the tonality will always be the same irrespective of the key in which it is played.

[Figures 4–6] compare values of the ratios to the note C in the Well-Tempered, Just Intonation, and the Pythagorean Scales. The corresponding frequencies are based on note A tuned at a frequency of 440 Hertz. [See Figures 4–6, summarized on Page 3 of this document.]

The discrepancies shown present definite problems when tuning fixed-tone instruments such as the harp or piano.

Conclusion

Without a doubt there is a correlation between mathematics and music which is supported from general observation, from historical considerations, and from a philosophical viewpoint. From a physical point of view, there are other aspects such as acoustical environments, vibrating membranes, air columns, and harmonic analysis. The feeling of an affinity is more than coincidental. Music is very much mathematical and mathematics is very related to music as shown from the construction of scale, the very basis of music.

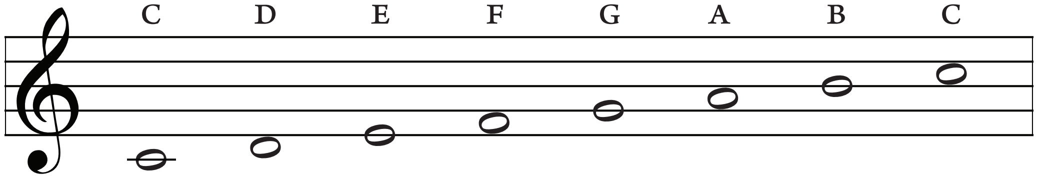
The mathematician J. Sylvester, in a footnote to his memoirs on "Algebraical research containing a disquisition on Newton's rule for the discovery of imaginary roots," stated the relationship very nicely. He wrote:

> **May not Music be described, as the Mathematic of Sense, Mathematic as the Music of reason? The soul of each the same! Thus the musician feels Mathematic, the mathematician thinks Music — to receive its consummation from the other when the human intelligence, elevated to the perfect type, shall shine forth glorified in some future Mozart — instinctly foreshadowed in the genius and labors of a Helmoltz!** [Archibald, p. 2]

How true this is. How true.

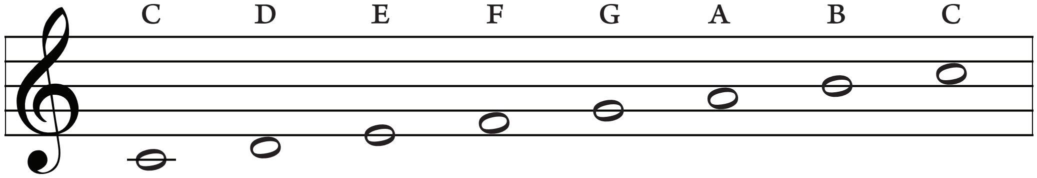
See page 3 of this document for Figures 4–6 and the bibliography.

Figure 4: Values of Notes in the three Scales expressed as fractionals.



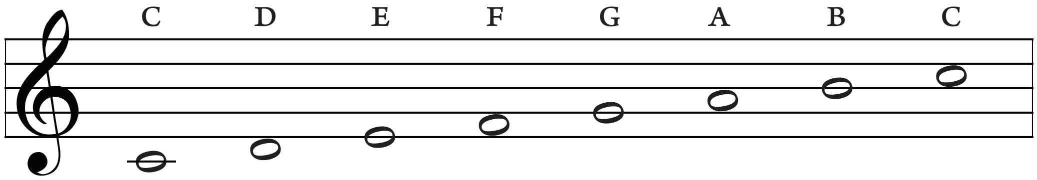
	Do C	*Re* D	*Mi* E	*Fa* F	*Sol* G	*La* A	*Ti* B	*Do* C
Pythagorean	1	1.12500	1.26562	1.33333	1.50000	1.68750	1.89843	2
Just Intonation	1	1.12500	1.25000	1.33333	1.50000	1.66667	1.87500	2
Well-Tempered	1	1.12246	1.25992	1.33484	1.49831	1.68179	1.88775	2

Figure 5: Values of Notes in the three Scales expressed as frequencies in Hertz, where the note A = 440 Hz.



	Do C	*Re* D	*Mi* E	*Fa* F	*Sol* G	*La* A	*Ti* B	*Do* C
Pythagorean	260.7407	293.3333	330	347.6543	391.1111	440	495	521.4814
Just Intonation	264	297	330	352	396	440	495	528
Well-Tempered	261.6258	293.6645	329.6276	349.2286	391.9966	440	493.8842	523.2517

Figure 6: Values of Notes in the three Scales expressed as ratios or powers of two.



	Do C	*Re* D	*Mi* E	*Fa* F	*Sol* G	*La* A	*Ti* B	*Do* C
Pythagorean	**1/1**	**9/8**	**81/64**	**4/3**	**3/2**	**27/16**	**243/128**	**2/1**
Just Intonation	**1/1**	**9/8**	**5/4**	**4/3**	**3/2**	**5/3**	**15/8**	**2/1**
Well-Tempered	2^0	$2^{(2/12)}$	$2^{(4/12)}$	$2^{(5/12)}$	$2^{(7/12)}$	$2^{(9/12)}$	$2^{(11/12)}$	$2^{(12/12)}$

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This document was remastered 24 January 2011 by Michael Thomas De Vlieger. The following were added: The photograph of Dr. Impagliazzo and caption, the summary Figure 6, and the red and semibold highlighting of text and figures. The two-part article, divided as stated above, is concatenated to read as a single article in this document. Dr. Impagliazzo's original Figure 4 (a spreadsheet-style table of fractionals and frequencies) is organized in Figures 4 and 5 in this document in the same fashion as Figures 2 and 3.

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