The Dozenal Society of America

Practical Polygons

by "Troy" (Donald Hammond)

Regular polygons have fascinated and influenced people throughout history. Few indeed do not respond in some degree to the symmetry and character exhibited by these shapes, each unique and yet plainly part of a family.

Straight-sided, they all yet inhabit a circle precisely — in fact, the circle is their final expression, the Nirvana of regularity — and the rotational and lateral symmetries thus ordained give the regular polygons not only their visible perfections of mathematical form but also the indispensably useful geometry which some of them afford to Nature and engineering alike. The equilateral triangle, the square, the regular hexagon, the circle itself — all combine purity with practicality. Selected regular polygons are quite fundamental to our comprehension of space and are hence perhaps worthy of more appreciation and study than is presently fashionable in education.

It is hoped, in this article, to emphasize the way in which regular polygons can be seen as two-dimensional expressions of numbers: pictures, as it were, which illustrate more clearly than does arithmetic the connections between numbers themselves and what we can perceive, measure, and use in the real world.

The Plane Truth

If what we see is to approach the actuality, it must be seen in two dimensions. While we may be impressed by the view of, say, the Parthenon when seen from an angle which shows it to be a solid and imposing structure, we are nevertheless obliged to move and look directly at one face at a time in order to discern the real shape: a viewpoint which gives the true angles and proportions of the front of the Parthenon renders its flanks invisible. We see but partially into the third dimension of space and optical illusions abound in line-of-sight perception (it is recorded that, during the building of Kilsby tunnel on the London and Birmingham railway, three men were killed as they tried to jump, one after the other, over the mouth of a shaft in a game of follow-my-leader[*]) and accurate perception is confined to the lateral plane.

Hence, despite the availability of various methods of pictorial representation, engineers still insist on three-view orthographic projection for their working drawings and demand auxiliary projections to show the true shapes of angled surfaces. Architects know that the true plan determines the building. Plane, two-dimensional figures are thus paramount to our understanding of space and proportion; and the regular polygons — from line-segment to circle — constitute a definitive basic set by which order may be perceived in, and structure imposed on, our surroundings (Fig. i).

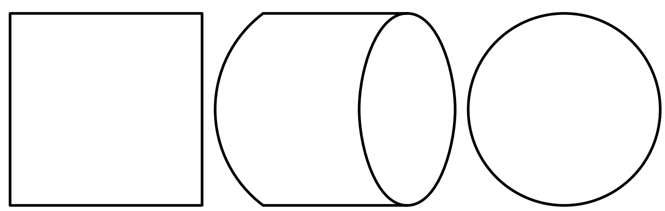


Fig. i

Not the least of these virtues is the insight given into the nature of numbers themselves: expressing numbers by presenting them as vertices of regular polygons shows very clearly properties which are not always evident from mere counting — particularly to children, but also to many adults who have never really understood numbers — and the educational need for such understanding will not, surely, be disputed in these days of declining numeracy?

$1 \longrightarrow \cdot$

$2 \longrightarrow 1 + 1 \longrightarrow \cdot^{\cdot} \longrightarrow \diagup$

$3 \longrightarrow 2 + 1 \longrightarrow \cdot\cdot^{\cdot} \longrightarrow \therefore \longrightarrow \triangle$

One is a dimensionless point. *Two* is the first *line* number: we need two points between which to perceive length, the first dimension. It is possible to

[*] See "The Railway Navvies" by Terry Coleman (Pelican A903).

think of two as a regular two-sided polygon which has length, but no area.

The difference between two and three is not merely one unit: it is the difference between a line segment with only one dimension and an equilateral triangle which has two dimensions. Three is seen to be the first *area* or *surface* number and we could indeed measure area in terms of equilateral triangles. Each angle of the R-triangle is two-thirds of a right-angle and the shape will bisect into two drawing-board set-squares with angles of $\frac{1}{3}$, $\frac{2}{3}$, and 1 right-angle, each having a hypotenuse exactly twice the length of the shortest side. (It is important to note that these natural and convenient fractions of the right-angle cannot be expressed by the Grade protractor, which is the basis of the decimal-metric system.)

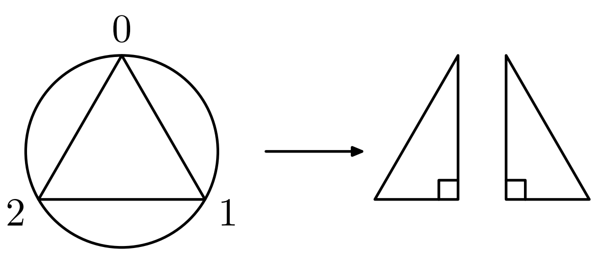


Fig. ii

Constructing the regular trigon in its native circle (Fig. ii) — easily done by children — shows also that with three, 'handed' rotation is possible: we can travel round 0 – 1 – 2 – 0 or 0 – 2 – 1 – 0, whereas two is simply an *alternating* number (cf. electric motors). From these easy beginnings, it already emerges that *three* is a number of considerable significance (well before any considerations of trigonometry). The trigon shows the character of the number, in a way that arithmetic cannot, so that it can be understood by even the youngest. Evidently, any number-system that takes insufficient account of the importance of three is going to run into difficulties quite rapidly.*

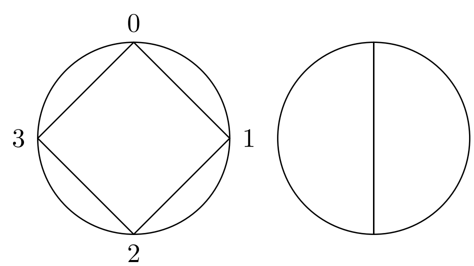
Four is the square number, and the square is the optimum shape for area-measurement (we could tessellate equilateral triangles or regular hexagons to reckon area, but they have lengths in three directions, whereas the square has only the two directions). The properties of this right-angle figure, fundamental to mathematics and just about every practical activity that exists, are generally well-known and need little amplification here. It is worth mentioning, however, that four is also the first *space* number: four points will delineate a tetrahedron and so contain a *volume*.

$$4 \to 2 + 2 \to 2 \times 2 \to \ldots \to \square \to$$ ⟁

So, the square of the first prime number has its own properties far in excess of mere evenness and is a foundation-stone of any number structure; yet, ten will not accept four as a factor. Twelve, of course, will. A bisected square gives the companion set-square, with half-right-angles.

Modular Figures

It was mentioned earlier that the characteristics of numbers could be displayed by polygons. This depiction is enhanced if *modular* figures are drawn: for each number n, circles are drawn with their circumferences divided into n equal parts; and modular shapes appear as, starting from a zero (top), we "jump" round the circumference by ones, then by twos, threes, and so on, joining successive points with straight lines until we arrive back at zero.



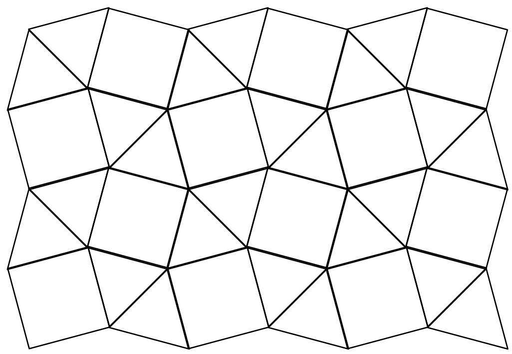
Mod. 4

Four thus gives a square (4/1) and a straight line-segment (4/2). The line-segment is, in this context, to be regarded as a two-sided figure and will appear with each *even* number.

With division of the circle into two, three, and four equal parts, we have obtained regular shapes which are, above all, *useful* — indeed, one may say, indispensable — in that they form an indissoluble bond between mathematics and practical necessity. Line, triangle, square: length; the basis for geometry, trigonometry, and engineering construction; the

*3 is the second prime number and, in conjunction with 2, controls the positions of all other primes: these are all members of the set $(2 \times 3)n \pm 1, n \neq 0$, which is the least set to contain them.

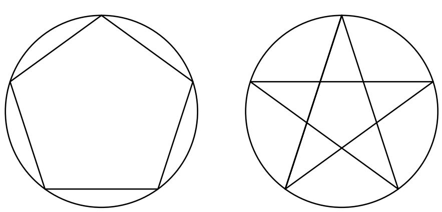
right-angle and optimum shape for areal mensuration. Not a bad score! Regular trigons and squares will tesselate in the plane with each other as well as with themselves —



— and, so far, all seems harmonious and clear.

Into the Shadows

Five is a number at the edge of darkness. It is the third prime number and, for practical purposes, the last one for which there is an exact construction. The construction is considerably more complex than hitherto, but is mathematically interesting: it involves division of a line-segment in Mean and Extreme ratio — the Golden Section — and will divide the circle to give a regular pentagon.[*] The shapes in the set are more suited to decoration and psychology than to engineering and mensuration and are much favoured for symbolic and magical purposes.



Mod. 5

The powerfully-attractive star figure, used in Heraldry, is 'good' when this way up, and 'bad' when inverted.

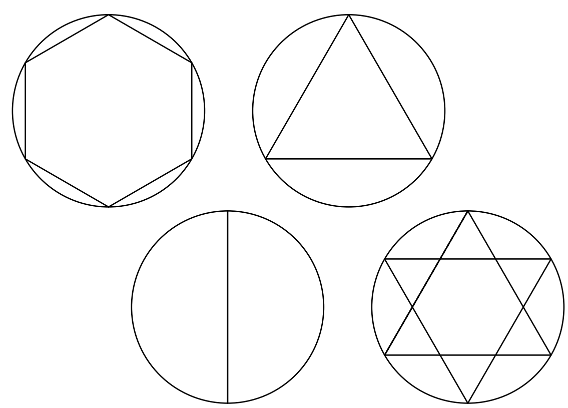
Five is also the number of digits on the human hand, and it is a matter for regret that the primitive resort to finger-count has allowed this otherwise rather impractical number to usurp the rightful place of three as the second prime factor of the counting base in general use.

[*] See Dozenal Review No. *30.

Regular pentagons will not tessellate in the plane with themselves or with other regular polygons (there is a three-dimensional relationship with a dozen: twelve regular pentagons form a regular dodecahedron), but, by comparison with the illumination given by two and three, five is a shaded and mysterious number: mathematically significant but of limited use. We should respect five and allow for its angular importance; but should not succomb to its esoteric charms when tackling arithmetic!

Daylight Again

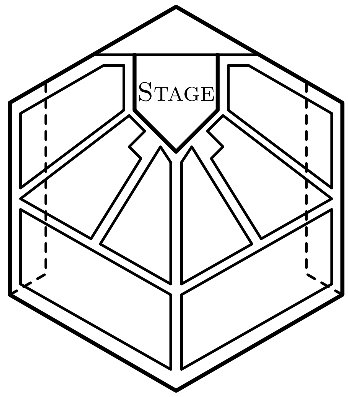
Six is the product of those all-important first two primes and gives us another useful shape: the regular hexagon. This is an *optimum*, and used as such by both Nature (honeycombs) and engineers (nuts and bolts). A plane tessellation of regular hexagons gives bees and wasps the most efficient possible accomodation for their pupæ; a square nut is weak and an octagonal one slips in the wrench, but the hexagonal nut is just right.



Mod. 6

It is amusing to squeeze a pack of cigarettes and watch the hexagonal tessellation form itself as the area available gets reduced!

The modular set for six exhibits the equilateral triangle and the line; six also controls the location of prime numbers and could make quite a good number base. The square is lacking, however, and so we must seek a little further for the best possible.

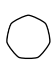


Chichester Festival Theatre

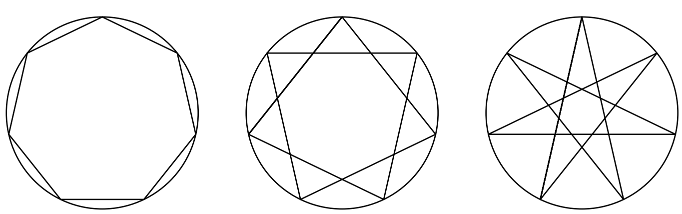
The optimal properties of the hexagon are not lost on some architects: the Festival Theatre possesses the presentational virtues of Shakespeare's "Globe" combined with the strength and economy of steel-frame construction using ordinary straight stock, and is a much-admired, successful threatre.

Seven and Eleven

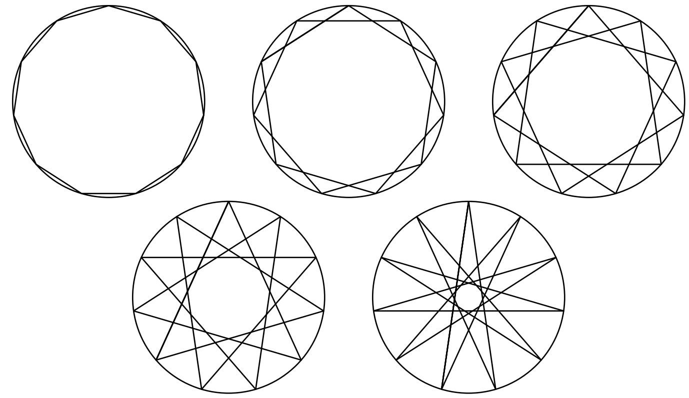
The gloom we saw gathering about five becomes fully established with primes greater than five as far as circle-division is concerned. It is not possible to find an exact construction for seven and eleven in the plane. The modular sets for these numbers show the pattern started with five: one convex polygon followed by a set of re-entrant star figures (dark stars?). A heptagonal curve-of-constant-width is used for coins by British decimalists: perhaps they hoped thereby to make such coinage interesting to people being deprived of the richness and beauty of £sd?



Decimal coin



Mod. 7



Mod. Ɛ

The seven-day week arose from the Lunar cycle and accounts for the inclusion of seven as a factor in the larger English weights, which proceed in a binary fashion up to 4 Lb, but then go to 7 Lb, so that all the ensuing weights up to 1 ton are divisible by 7 as well as by 2. It makes sense if one is feeding animals or firing boilers, etc., and wishes to order fuel or food at a regular time in the working week, since the arithmetic is then easy.*

Three Curate's Eggs†

Eight, nine, and ten are good in parts: the modular set for eight gives the square and a line-segment; nine gives the regular trigon; ten (decim) yields a regular pentagon, a line, and two stars. Eight thus shows two useful shapes, but there is little to choose between nine and ten on this criterion: each has two re-entrant figures and one useful shape. Admittedly, two is a more important prime factor than three; and ten accepts two. The overall weakness of the decimal base is, however, shown very clearly by this two-dimensional analysis: running one's gaze along the modular set for ten shows at a glance the paucity of useful relationships afforded by this number.

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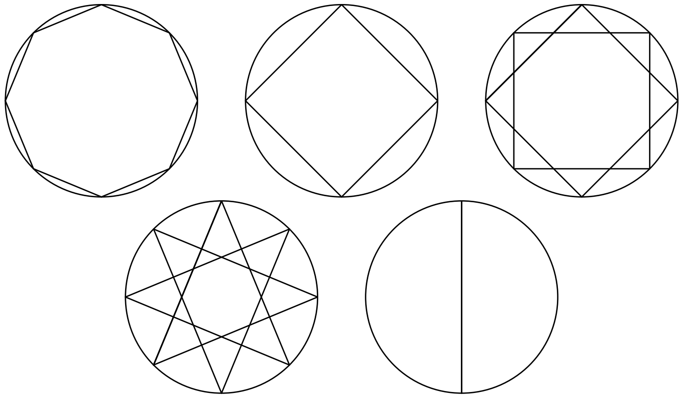
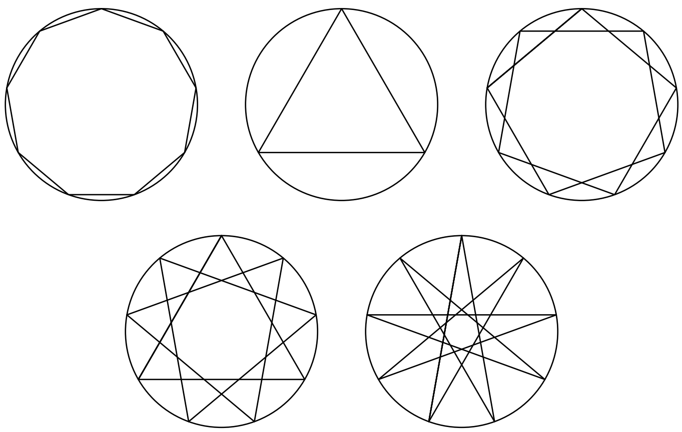
* The English ton is decimally 2240Lb, dozenally 1368Lb and is divisible by two, five and seven. 2 Lb per day = 1 st./week.

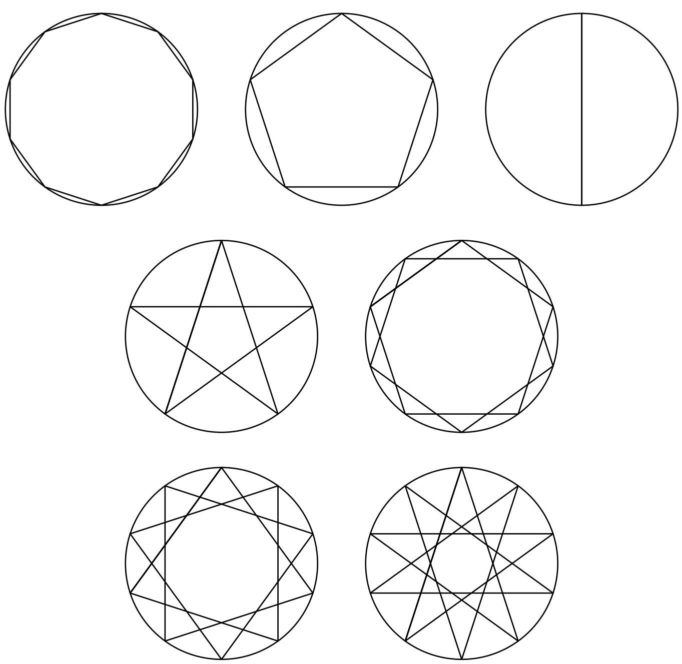
† A British term describing something that is mostly bad, but has good qualities which are endowed with disproportionate redeeming effect. —Ed.



Mod. 8



Mod. 9

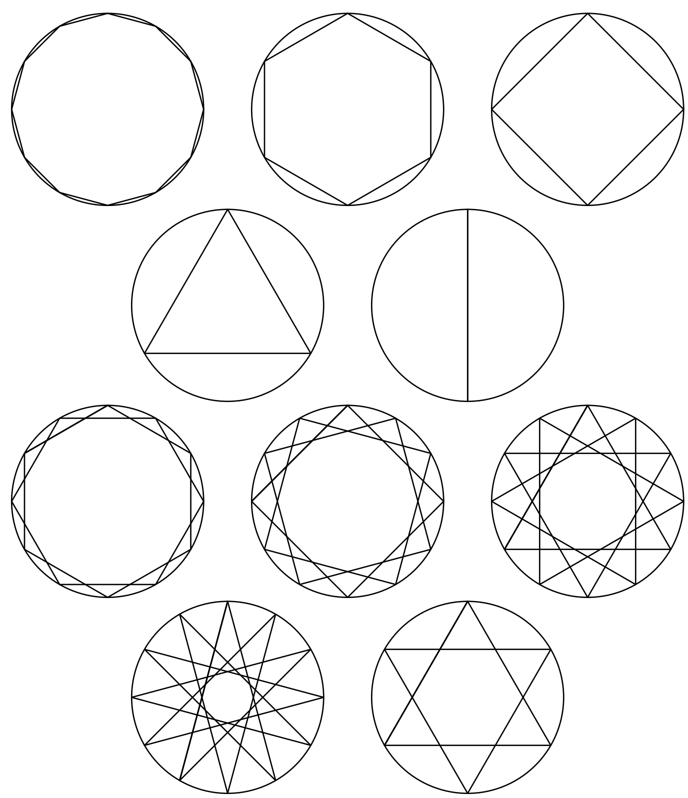


Mod. ↊

Arrival

As we proceed, regular polygons with increasing numbers of sides grow more and more to resemble the circle — which is indeed a regular polygon with an infinite number of sides — and to lose their very angularity. It has, however, been well worth coming this far! Twelve is really a 'box of delights': the modular subjects of the regular dodecagon include those very shapes we know to be essential; and the dozen is the least number to do so.

The offspring of the dozen serve us well. Five of the six possible figures are convex polygons and four of these are essential to engineering and mathematics. If we omit the regular dodecagon itself (and it has significant properties, of which more in a later article), we get a ratio of four essential shapes to six possible. This two-thirds ratio is exceeded by six, which gives unity in this respect: *all* the shapes for six are basic; but six will not provide the necessary square, and so twelve is the key number.



Mod. 10 (1 Dozen)

Need we search any further for a rational, serviceable number-base? *Can* there possibly be a better?

* * *

The table below compares the modular-plane-figure content of each number up to one dozen.

Circle divisions	No. of figures in the set	No. of convex figures	No. of essential shapes
2	1	1	1
3	1	1	1
4	2	2	2
5	2	1	0
6	3	3	3
7	3	1	0
8	4	3	2
9	4	2	1
↊	5	3	1
↋	5	1	0
10	6	5	4

The presence of a prime number is shown by a "1" in the third column.

It can be argued that, apart from the circle itself, there are only four essential regular polygons for practical purposes. Twelve has them all.

* * *

In part one of this article it was observed that modualr division of the circle gives rise to families of regular polygons, each polygon being thus a two-dimensional pictogram showing the characteristics of a number more clearly than is possible with a mere symbol (the equilateral triangle really is a *picture* of three). It also emerged that the modular shapes generated by the number twelve are those very polygons whose angular properties underly both engineering and natural structures; thus pointing to the significance of twelve as the most efficient base number.

More About Tessellation

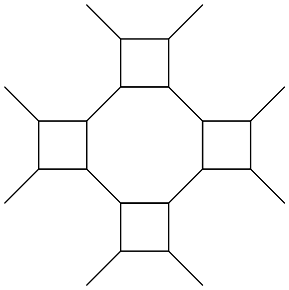
No apology is made for returning to this topic: tessellations of regular polygons illustrate very well how numbers work with each others — or themselves — in two (or more) dimensions.

Purely regular plane tessellations (all tiles the same shape) are well-known and use only regular trigons, squares and regular hexagons (3^6, 4^4, and 6^3). No others are possible: for a plane tessellation, the interior angles of polygons meeting at a vertex must sum to exactly one turn, or four right-angles. (The notation used above is that employed by Messrs. Cundy & Rollett in their standard work: 'Mathematical Models', whereby the cardinal numbers indicate the polygon(s) concerned and the indices denote how many of each meet at any vertex in the tessellation. Examples: 3^6 means that six equilateral triangles meet at a vertex, and 4.8^2 means that one square and two regular octagons are contiguous in this way.)

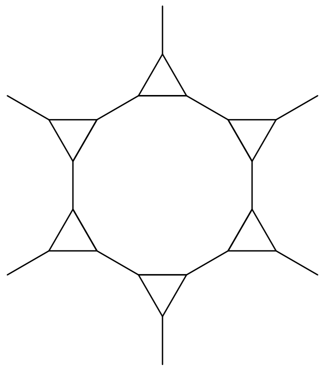
The *semi-regular* tessellations consist of mixtures of regular polygons to cover the plane. There are just eight of these, and their notations are:

$3^2.4.3.4$, $3.6.3.6$, $3^4.6$, $3.{*}10^2$, 4.8^2, $4.6.{*}10$ and $3.4.6.4$.

Note that *10 means one dozen! Two of these are illustrated overleaf[*]; those so inclined may enjoy constructing the set...



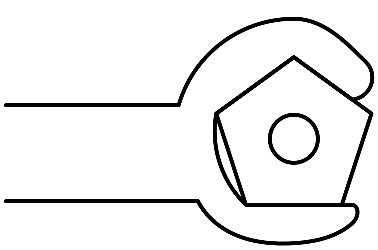
4.8^2



$3.{*}10^2$

Eight semi-regular tessellations, together with the three regulars, are the *only* plane tilings possible using regular polygons as elements. Again, the nature of just which numbers work together in such harmony is clear: they are simply related to the dozen; and the twelve-sided dodecagon is the last and greatest regular polygon which can be used in such patterns.

It may be noted at this point that regular PENTAGONS and DECAGONS cannot form plane tessellations — by themselves, with each other, or with any other regular polygons — and an extra dimension is needed in order to accommodate these five- and ten-sided shapes. This is something to be borne in mind by those who advocate adoption of the foundation-stone of the decimal-metric system: the Grade protractor. What will their draughtsman's set-squares be like? The present "thirty-sixty" set-square won't fit a centesimal angle-scale; so, shall we see pentagonal nuts and bolts? Spanners and wrenches with one flat jaw and a corner opposite? Difficult to slide-on, I should think!



Spanner for decimal nuts...?

We can quite easily predict which tessellations are likely to work by making a list of the sizes of interior angels for regular polygons, *in terms of right-angles*:

Eq. triangle	$\frac{2}{3}$	R. octagon	$1\frac{1}{2}$
Square	1	R. decagon	$1\frac{3}{5}$
Pentagon	$1\frac{1}{5}$	R. dodecagon	$1\frac{2}{3}$
Hexagon	$1\frac{1}{3}$		

Knowing that each plane vertex demands exactly four right-angles, and that all vertices must be congruent, we look for collections of the above rationals which add-up to four. For example:

$1\frac{1}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 4$, which shows that regular hexagons will tessellate (which we knew anyway); but

$1\frac{2}{3} + 1\frac{1}{3} + 1 = 4$ is more interesting and suggests that a tessellation of regular dodecagons, regular hexagons and squares is possible (and indeed it is). In some instances, however, a collection which sums to four will not extend to a full tessellation because the vertices cannot be made congruent throughout. It is just this which goes wrong when we try the pentagon and decagon:

$1\frac{1}{5} + 1\frac{3}{5} + 1\frac{1}{5} = 4$, but will not tessellate in any arrangement... Poor decimal! So near and yet so far! Here, ten will not work even with its own factor, five, whereas the polygonal factors of a dozen — three, four, six, and twelve — harmonize elegantly; even eight is used in one instance.

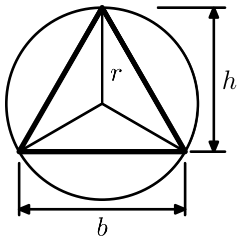
Areas of Regular Polygons

The calculation of area affords further insight into how the regular figures illuminate the essential tractability of twelve-based shapes (and numbers) by comparison with the denary set. Any regular polygon, of course, has by definition a circumscribing circle and — *if the polygon is constructible in the plane* — a surd[†] formula can be derived, giving its exact area in terms of r, the radius of the circumscribed circle.

For example, the area of an equilateral triangle (regular trigon), drawn in a circle of radius r, is found as follows:

[*] In the original; in this edition they are directly below. —Ed.

[†] That is, a formula that cannot be so simplified to remove any square root expressions. —Ed.



$h = \frac{3}{2}r$ (const.), $h = \frac{2}{\sqrt{3}}h$ (Pyth.)

$\therefore b = \frac{2}{\sqrt{3}} \cdot \frac{3}{2}r = \sqrt{3}r$

Area of equilateral triangle
$= \frac{1}{2}bh = \frac{1}{2} \cdot \sqrt{3}r \cdot \frac{3}{2}r = \frac{3\sqrt{3}r^2}{4}$

So, we have a fairly simple formula with one square root. The area of a regular hexagon in a circle with radius r is twice that of the equilateral triangle:

$$A_{\text{hex}} = \frac{3\sqrt{3}r^2}{2}$$

The regular hexagon turns up quite often in engineering and nature; it also possesses the unique property that its length of side, L, equals the radius of its circumcircle, r. The length of side is easily measured and so here, for those who may find it useful, is a play-on-word mnemonic for the area formula:

The areal bee
  Makes hexagons true:
Each three-root-three
  L-squared over two…

The regular octagon features, not surprisingly, root-two:

$$A_{\text{oct}} = 2\sqrt{2}r^2$$

The square is, naturally, even simpler:

$$A_{\text{square}} = 2r^2$$

However, the most impressive (and satisfying) result is the area, in these terms of the regular dodecagon:

$$A_{\text{dodec}} = 3r^2$$

— just that! Our twelve-sided regular polygon, which underlies all those useful subset shapes (see Part I) and tessellations, and whose benign geometry defines our clock-dials, shares only with the square itself the distinction of a completely rational area (there can be no others.

So, what of five and ten? Regular pentagons and decagons can be constructed in the plane and their areas, too, can be expressed in surd form in terms of r. But what expressions they are!

$$A_{\text{pent}} = \frac{5r^2}{4}\sqrt{\frac{5+\sqrt{5}}{2}}$$

$$A_{\text{dec}} = \frac{5r^2}{4}\sqrt{\text{↊} - 2\sqrt{5}}$$

In each of these we get a square root of a square root — a fourth root. Just as the figures themeselves need an extra dimension for tessellation, so do they require a second-stage irrational for area calculation…

Rational		Square Root		2nd and 4th roots	
Square	$2r^2$	Trigon	$\frac{3\sqrt{3}r^2}{4}$	5-gon	$\frac{5r^2}{4}\sqrt{\text{↊} - 2\sqrt{5}}$
10-gon	$3r^2$	6-gon	$\frac{3\sqrt{3}r^2}{2}$	↊-gon	$\frac{5r^2}{4}\sqrt{\text{↊} - 2\sqrt{5}}$
		8-gon	$2\sqrt{2}r^2$		

Conclusion

This two-part article has considered aspects of certain numbers as revealed by depiction in the form of modular polygons. We see most clearly in *two* dimensions and this representation allows insight into the nature of low numbers deeper than that afforded by linear arithmetic. What emerges very strongly is the extraordinarily pivotal role of twelve as the framework for those polygons which are both readily constructed and indispensable in the practical world. Practical polygons, it seems — like practical measurement and practical arithmetic — come, as it were, by the dozen!

This article was originally published in Numbers 4 and 5 of The Dozenal Journal, which was a joint publication of the Dozenal Society of Great Britain and the Dozenal Society of America. Number 4 was released in the spring of 1196 (1986.) in Denmead, Hampshire, England. Number 5 was released in spring of 1197 (1987.) in the same place. Donald Hammond, a long-time stalwart of the DSGB, published this article (along with many other excellent pieces) under the pseudonym "Troy."

The work has been completely retypeset using the LaTeX document preparation system, and is here set in Latin Modern, 12×15. The figures have all been redesigned in the METAPOST graphics description language. In addition to this, the following alterations were made: Oxford commas were added throughout. Also, ellipses have been made consistently three dots throughout, except at the end of a sentence, when they are four dots. Variable names in equations, such as r, have been consistently italicized. On page 2, variable "N" has been rendered n. In the footnote on page 3, "Dozenal Review" was changed to small caps, and the footnote marker was placed *after* the period. On page 4, for the footnote about the ton, the footnote marker was moved to *after* the period. On page 4, the image of the hexagon containing what appears to be a Greek warrior's picture was omitted. The theater, on the other hand, was redrawn in METAPOST. On page 4, "elf" has been replaced by "eleven." On page 4, another reentrant star was added for mod. 8. On page 5, the order of presentation for the modular figures for 9 was changed, and another star was added. On page 5, two more shapes were added for the modular figures for Ɛ. On page 5, the order of presentation for the modular figures for 10 was changed, and several more shapes were added. On page 6, the typographical error "emplyed" was fixed to "employed." On page 7, the table of numbers of right angles in various polygons was completely redesigned. On page 8, in the phrase "regular pentagons and decagons can be constructed," the first word was capitalized. On page 8, the table was extensively redesigned.

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