The Dozenal Society of America

Boxes & Cans

Some Points on Packaging*

by Troy

From the beginning of his existence, man has had to recognize constraints; in the early days the need for food, coupled with the fleet-footedness of mobile dinners and the fearsomeness of equally mobile diners made it clear that humanity was definitely out of the Garden of Eden, and that brains were going to be needed from now on.

Out of the Garden of Eden...

Whatever muscle-power Ug, the cave-man, might possess; however splendid an athlete he might be, there was always a specialized animal (be it one which he wanted to catch, or that wanted to catch him) which could do it better! (Many have forgotten all this, nowadays; perhaps those who strive to run and jump ever faster and higher should contemplate the effortless progress of a cheetah at forty miles an hour, or the precision of [a] domestic tabby casually leaping on to a narrow wall four times her own height...)

We must learn Nature's rules, whether we like it or not; it is only by recognizing constraints that we survive and, paradoxically, achieve a degree of freedom and security. Just as Ug learned to make traps and spears to avoid wasting his limited muscle-power, so we have to understand how to avoid wasting our materials and resources in a much more complex world.

It is certain that some of the more subtle rules began to be apparent with the advent of early farming: domestic animals needed to be penned-up for the night with expensive hurdles or stone walls; land had to be divided without waste; grain, beer and preserves needed storage and, of course, dwellings had to be built. These activities were the first examples of *packaging* and called forth a whole new set of rules which govern the 'inanimate' Universe; and which are far more rigorous than those Ug used for dealing with his animals. After all animals can be frightened by fire and even quite intelligent humans can be scared out of their wits by a little applied theatre — the behavior of living things can be altered — but no amount of ritual, spear-throwing, wearing hideous masks or even counting in tens is going to make the slightest effect on the calm mathematics of space and time. While Winston Smith, in extremis, might find it possible to convinced himself that $2 \times 2 = 5$ and many today believe that, say, 0.25 is absolutely the same thing as one quarter, such beliefs are not

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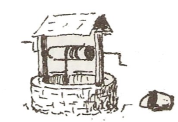
wholly sustainable and tend to distance people from reality. Nature does offer us clear rules for mathematics, which cannot be broken and with which we may choose to live in harmony, or not: in no way can we change the rules! Some find this rather depressing, particularly when hastily-adopted notions turn out to be false, but most of us should find such constancy reassuring: it is so pleasant to find something which just cannot be changed at the whim of politicians, is it not?

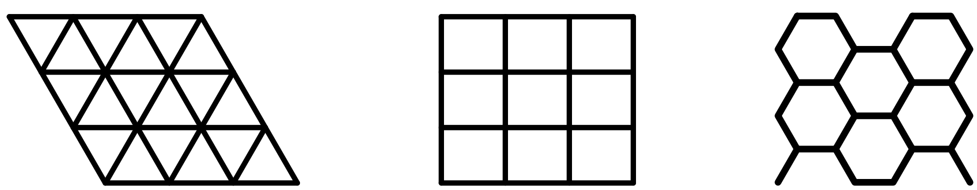
Shapes: Area, Perimeter & Tessellation

The circle gives the maximum area which can be enclosed by a given perimeter. Thus, if someone has a length of fencing and wishes to enclose as many animals as possible, then a circle is the shape to aim for: a dozen hurdles arranged as a regular dodecagon[†] will pen-up more sheep than will the same number used as a 3×3 square, because the former shape is closer to a circle (the square has an area of $9u^2$, the regular dodecagon over $\mathcal{E}u^2$).

Circles, however, will not *tessellate*: however closely we pack them together, there will be gaps. So, although many things are best made round — lighthouses, windmills and covered-wagon defences against enraged Red Indians — we have to use angular polygons to cover a surface completely, be it a patchwork of fields or a pavement. Such tessellations may be regular, compound, uniform or random, but all are polygonal. If, as in the case of the pavement, the separate elements have to be *made*, then it is convenient and cheap to have them all the same shape: ordinary uniform brick walls and pavings are the chepaest kind and involve the least effort in assembly, as every householder who has tried his hand at crazy-paving knows (during the process, he learns the real reason for calling it 'crazy'!).



Many things are best made round...



There are three, and only three, regular plane tessellations: equilateral triangles, squares and regular hexagons (3, 4 and 6 sides: which number-base copes with these readily...?). These three shapes will also tessellate with each other to form semi-regular arrays, of which there are eight.[‡] Of course, there are non-regular shapes which will tessellate (an infinity of them) and this enables us to please ourselves in all matter of pavements and other flat tilings; however, when it comes to putting things in *boxes*, more constraints — of a pressing economic character — appear.

Boxes, made of cardboard, wood, etc., are most easily and cheaply made cuboidal in shape, because of the flat nature of the materials; and, happily, cuboidal boxes will tessellate in *three* dimensions (stack properly), which is necessary to avoid wastage of space in rectangular

[†] An elegant property of the regular dodecagon, drawn in a circle of radius r, is that its area is exactly $3r^2$. This is not made much of in these decimal days.

[‡] For a full and most interesting dissertation of all aspects of tessellation the reader is referred to "Mathematical Models" by Cundy and Rollett (Oxford University Press).

warehouses, lorries, etc. Thus, non-perishable goods like sheets of paper, car spares and woodscrews can be packed in small boxes within larger boxes with little difficulty.

Problems, however, arise with liquids or perishables which must be in *sealed* containers: these vessels come in the class of things which are best made round (or at least cylindrical). While it is possible to make cans of square section, it is not easy; and, bearing in mind the area/perimeter property of the circle, wasteful. (Certainly, *some* liquid containers, — gallon cans — are made in a rectangular shape very often — though never paint cans, for reasons which are obvious — but that is a particular answer to a particular problem.) Our humble cans of beans, our wine bottles and, for that matter, the bodies of bee pupæ, are essentially cylindrical, so optimizing ease of making, simple sealing and economy of materials. We then pack these cylinders into rectangular boxes!

It may be interesting briefly to consider how Nature tackles the problem: bee pupæ are packed into regular hexagonal cells, which is an *evolved* ideal. The hexagonal prism is the most economical possible shape to meet the conflicting demands of containing the round body of the pupa and of tessellating with its companions. There may at one time have been bees which used square cells; they would, however, have been quickly ousted by the more numerous hexagon-users, which could hatch-out more bees per unit of expensive wax used in the ratio $2\sqrt{3} : 3$, or about $7 : 6$. We cannot use this elegant solution because the hexagons do not pack properly into rectangular frames (honeycombs are designed to *lodge* in irregular spaces).

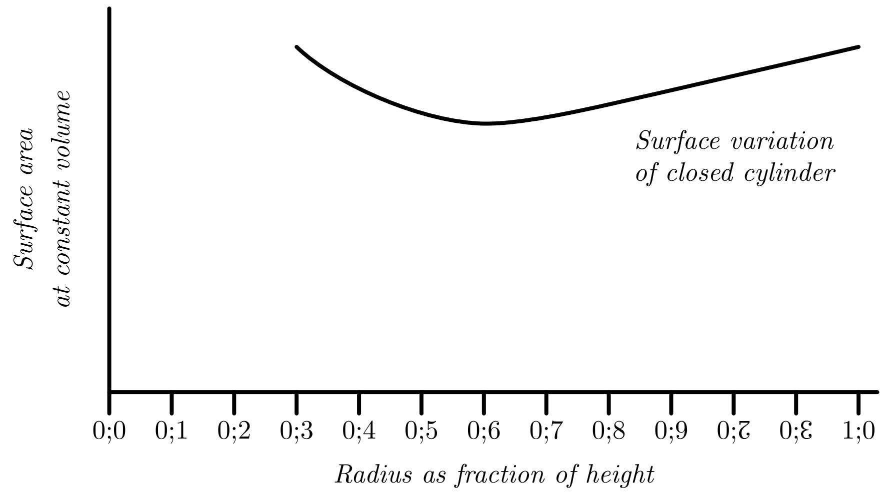
Thus we are faced with an economic imperative: round containers in rectangular boxes! (Is there a philosophical point lurking here somewhere which should be considered by those who love rigid uniformity?) Before we look at how this should be arranged, however, we must take note of certain constraints on the dimensions of the cans themselves.

The Optimum Tin Can

As we increase the height of a cylindrical can whilst holding the diameter constant, the contained volume increases in direct proportion, as does the area of metal in the curved surface. Keeping the height constant and inreasing diameter gives an increase in volume proportional to the *square* of the diameter, whereas the metal used then increases both as the square of the diameter (for the ends) and linearly as the diameter (for the curved surface). There is thus an *optimum* ratio of height to diameter for a cylindrical can, which gives the least consumption of metal for a given volume. This is shown on the graph below, and is calculated at the end of this instal[l]ment.[§]

While there are other factors, some of them psychological, which affect the proportions of cylindrical cans, it is apparent that the cheapest form is a can whose diameter equals its height. Such a can would occupy a cube of space in a rectangular box and so it is easy to reach a first approximation of the area of cardboard which must be used to make an enclosing box. In the following section, which looks at various can-packing arrangements, I have ignored the double-top and double-bottom, and simply noted the basic enclosure area for a box, using one can-diameter as the unit of length.

[§]For this edition, which is published with both parts together, this mathematical demonstration has been moved to the end of the article. —Ed.

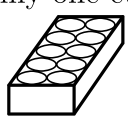


Tens and Dozens

Basic areas of cuboids for enclosure of optimum cylindrical cans. 1 can dia. = 1 unit.

a) *TEN cans.* (Only one cuboidal arrangement possible.)

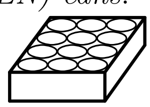
$1 \times 2 \times 5$



Total area needed = *2↊ u²
Area per can = *3;4↊ u²*

b) *TWELVE (ZEN) cans.*

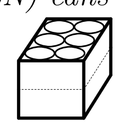
$1 \times 3 \times 4$



Total area needed = *32 u²
Area per can = *3;20 u²*

c) *TWELVE (ZEN) cans* (with a dozen, two layers are possible)

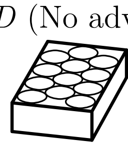
$2 \times 3 \times 2$



Total area needed = *28 u²
Area per can = *2;80 u²*

d) *HEX-NESTED* (No advantage when a rectangular box is used.)

$1 \times 4\frac{1}{2} \times (1 + \sqrt{3})$



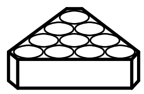
Total area needed = *32;↋7 u²
Area per can = *3;30 u²*

Packing in dozens shows an immediate advantage in the first two examples, where both are in single layers: the cost per can (or other object) of cardboard increases by more than ↊ per gross (over 7 per cent in decimal terms) by changing from dozens to decimal packing. This is an example of how disregarding natural constraints for political reasons can cause unnecessary waste and, of course, inflation. Decimal packing, where it occurs, is an attempt to adjust to decimal money; yet did we not once have twelve pence to the shilling. . . ?

The really decisive example is the two-layer form (allowed by the factorability of the dozen) in which the *total* enclosure area is less than the requirement for ten: rather than use method (a), which fits ten, it would actually be cheaper to use box (c) and leave two empty spaces!

Is there any hope for ten? Not a great deal. However, for those who are fanatically devoted to finger-count, here is a suggestion. As the Greeks tell us, ten is a *triangular* number, and so ten cans can be hex-nested to form an equilateral triangle of cans, which can then be packed into a triangular box! There is a little gain by doing this, which is enhanced if the corners of the box are truncated to give a sort of hexagonal prism with alternate long and short faces:

e) *TEN cans.*



Total area needed	=	*26;8Ɛ u^2
Area per can	=	*3;0Ɛ u*^{2}

This is still not as economical as box (c), but uses the least carboard possible to pack *ten.* It is offered in all seriousness to the denary brigade; but I think that, although such boxes would pack together quite well, their manufacture might be expensive. Moreover, those angles do not fit well with the Grade-protractor, do they?

As most readers will already have noted, the high efficiency of box (c) is given by its near approach to a cube; this is not the whole reason, though: one can in its own cubical box uses 6 square units of card, and 8 cans would need 3 u^2 per can. This latter is cheaper than the best possible with ten; but it is not as good as the dozen-pack! This point, together with a few more aspects of packaging, will be dealt with in the second part of this article.

* * *

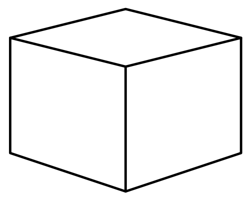
In the first part[] of this essay we looked at some of the basic facts about packaging — both large- and small-scale — and examined some reasons behind the facts. It was noted that packaging is a necessary adjunct, not only of human civilization but also of life itself. We noted, too, inexorable economic forces which come into play to determine *shapes* of packages, together with dimensional and numerical relationships available to meet these economic needs.

It will be recalled that metal cans need to be cylindrical and that their optimal dimensions are: height = diameter. It was shown that such cans are so much more cheaply packed by the dozen than in tens that a twelve-pack with two empty spaces actually costs less than a ten-pack completely filled! The question then arose: is the twelve-pack, with two layers of six, better than the decimal box because it is more like a cube? It was seen that an eight-pack could be a perfect cube, but is still less efficient than the dozen-pack.[]

Cardboard Costs		
No. per pack	Format	Units^2 per can
Twelve	$2 \times 3 \times 2$	2;80
Ten	Triangular	3;0Ɛ
Eight	$2 \times 2 \times 2$	3;00

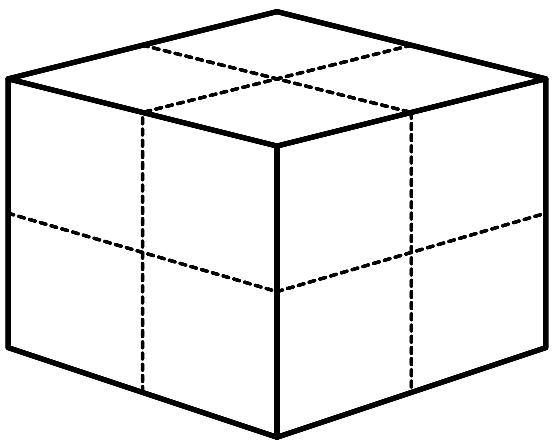
Squares and Cubes

Of course, a cube is the optimum shape for a cardboard box. The cube uses the least possible amount of flat card to contain a given volume. The reason why twelve-packs are cheaper than are eight-packs even though not quite cubical lies with another of Nature's rules: the *square-cube* law. If, for example, we have a cubical box containing one cubic unit of space, its surface area will be six square units.



V = 1 u^3
A = 6 u^2

So, a single can in its own little box uses six units of card. A larger cube (twice the dimensions, say) uses two dozen square units of card, but contains eight cubic units of space.



V = 8 u^3
A = *20 u^2

In this way, eight cans in a box use card at the rate of 3 square units per can — only half the card-cost of one per box.

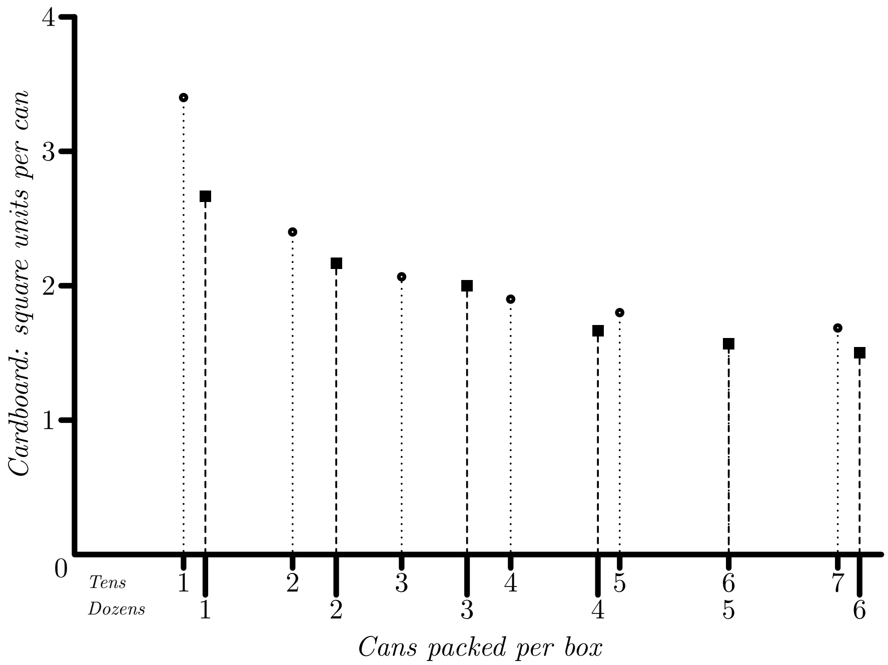
Going to extremes is sometimes instructive: the cost of card to contain twelve gross cans would be only *half a square unit* of card to each can. (Of course, one would need a fork-lift truck to carry such a box, and the box would certainly burst anyway; but the principle is clear enough: use the biggest boxes practicable!)

*   *   *

The SQUARE/CUBE law, stated rather loosely, is that as a body increases in size, *whilst maintaining the same shape*, its volume increases more rapidly than does its surface area. There is a graph at the end of this article to illustrate the comparative rates of growth.

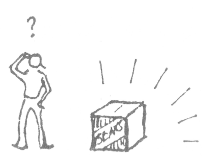
*   *   *

We now see that the dozen-pack beats the decimal for two reasons: it can be made closer to the ideal cube-shape, and it is *bigger*. The same twelve-pack beats the perfectly cubical eight because it is bigger; the eight pack beats the decimal box because it is cubical, though smaller — and so on. A diagram seems appropriate at this stage, to compare the economic merits of packing in tens and twelves. In every case, the box is assumed rectangular and cans packed in the most economical arrangement possible.



Comparative costs of packing by twelves and tens.

The diagram shows the general comparison for reasonable numbers of cans packed. Clearly, since boxes have to be (a) strong enough and (b) carried by men, the weight packed must not be excessive. One would need to pack one-and-a-half gross in one box to get the cost down to one square unit of card per can — and that might weight upwards of ninezen pounds, or roughly the same weight as a bag of cement: even if the box did not burst, there would be widespread lumbago among supermarket staff!



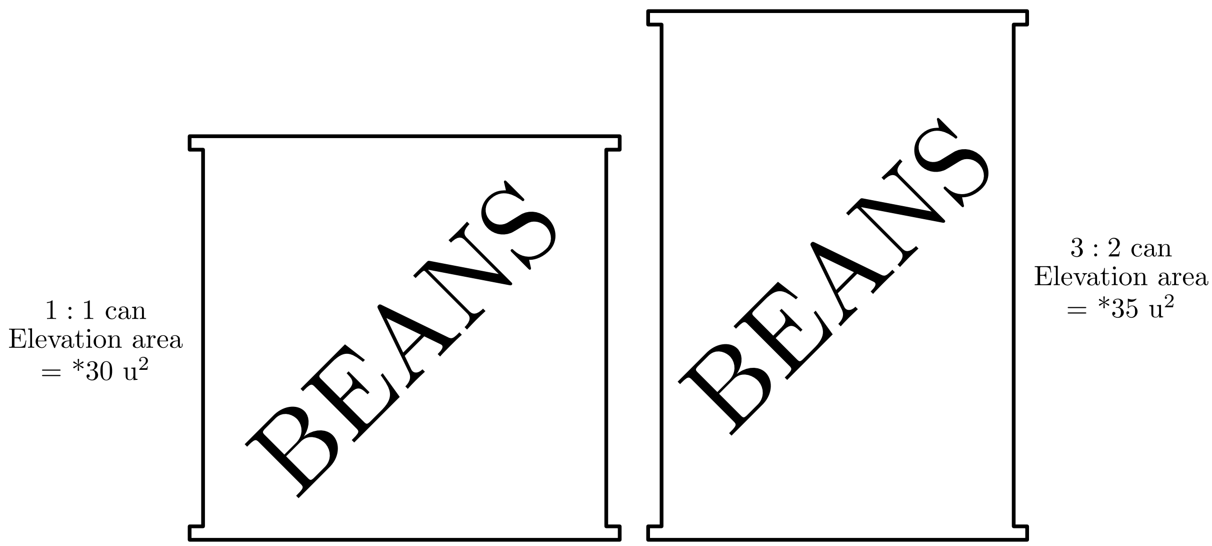
Proportions

The foregoing estimations have been based on cans with the proportions: height = diameter. This is the most economical ratio; yet many cans are not made so. Why is this? It is easy to note that the practical need to form raised seams at top and bottom of a cylindrical can — using metal whose penalty is greater for the circular ends than for the cylindrical body — may call for a slight adjustment favouring height against diameter; but many cans exceed this modest requirement and have height:diameter ratios ranging from 9 : 5 down to 1 : 2. A frequently-met profile is 3 : 2 (e.g. Heinz).

Evidently, there are further economic or practical aspects of marketing canned goods which justify departure from the optimum 1 : 1 ratio of height to diameter. For example, a shallow, wide can may be used for foods which are largely solid (tuna, salmon) and expensive: the shallowness makes for easier insertion and removal of the contents — which are more attractive in large portions — and a wide can looks more generous than does a rather tiny 1 : 1 version! Readers will doubtless find many more examples of such ‘marketing factors’.

The 3 : 2 Can

One particular style of tin can is worth a mention here: the very popular and widespread 3 : 2 type, in use for baked beans, soups, etc. Here are elevations of two cans (they are of equal volume).



The can on the left is cheaper, of course. The one on the right, however, despite holding no more, displays an elevation of greater area — it *looks* bigger. Furthermore, the 3 : 2 ratio occurs in the Fibonacci series:

$$1, 1, 2, 3, 5, 8, 11, 19, \ldots$$

—and so approximates to the Golden Section ratio which gives the rectangle whose proportions are held to be the most pleasing psychologically (compare with the dimensions of standard cine or still photographs).

Moreover, a telling practical consideration for this can is that a half-size version is feasible simply by halving the *height*. If we reflect that it is simple to cut cylindrical tube during manufacture to any length and then to fit standard end discs, whereas it is very expensive to make different tube diameters, then the point is clear. The 3 : 2 can and its half-size companion are *each* more costly than the 1 : 1 optimum[] but this extra cost is more than offset by the economy gained by use of a uniform diameter for both sizes.

(Incidentally, the use of 3 : 2 cans confers no advantage to decimal packaging: a dozen such cans in a box need an area of 3;6 square units of card per can, but the ten-pack requires 4;1 units. . .)

*   *   *

This concludes the present two-part article in which some principles of packaging have been explored. We have seen the need for packaging in general, for recognition of constraints — both mathematical and practical — on the design of packages and, arising from these, reasons why a great deal of commercial packaging is done in dozens; this last fact ensuring that the dozen will remain as a modular number, whatever its opponents may wish or do! It is certain that readers will find points I have omitted or failed to develop, and I hope they will write to the Journal with items of interest on the subject.

Appendices

Optimum Cylindrical Can

The volume of a closed cylinder is given by:

$$V = \pi r^2 h$$

while its area is given by:

$$A = 2\pi r^2 + 2\pi r h$$

We can arrange these to query the optimum height for a given volume, and substitute that into the equation for area:

$$h = \frac{v}{\pi r^2}$$

$$A = 2\pi r^2 + \frac{2\pi r V}{\pi r^2}$$

$$A = 2\pi r^2 + \frac{2V}{r}$$

Taking the differential of area with respect to radius:

$$\frac{dA}{dr} = 4\pi r - \frac{2V}{r^2}$$

But we know that $V = \pi r^2 h$, so we substitute that into the equation:

$$\frac{dA}{dr} = 4\pi r - 2\pi h$$

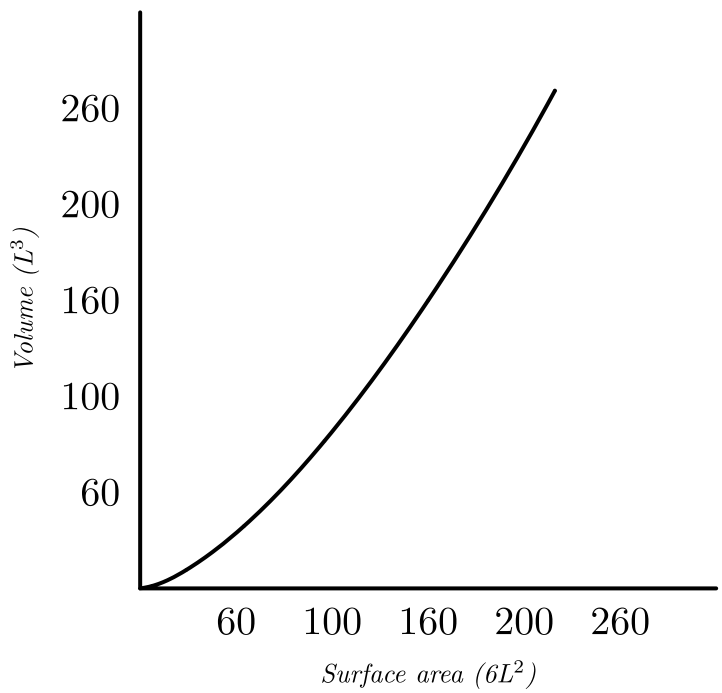
For stationary point $\frac{dA}{dr} = 0$, we solve:

$$2\pi h = 4\pi r$$

$$h = 2r$$

Noting that $\frac{d^2A}{dr^2}$ is positive, the stationary point is a minimum; *hence surface area is at minimum when height = diameter.*

The Square/Cube Law



Curve shows comparative rates of growth for surface and volume for a cube with edge length = L.

This document was originally published in volumes 2 and 3 of The Dozenal Journal in 1193 and 1194, in two parts. This remastered version was prepared by Donald P. Goodman III in 11↋9, using the LaTeX document preparation system and Metapost for redrawing the diagrams. The following changes were made in its preparation: 1. Several typographical errors were corrected, and some unnecessary ellipses were removed. These are marked by brackets in the text. 2. The format, though not the content, of the "Cardboard Costs" table was significantly changed. 3. The drawings were scanned and minimally improved for pdf publishing; specifically, the background color was set to a pure white. 4. The diagrams were all redone, in Metapost. 5. The proof of the optimal dimensions of a can was substantially reformatted, and the description of each step was expanded. 6. Hammond digit tens were replaced with Pitman digit tens (↊); these are quite similar to one another. 7. Various formatting changes were made to suit the article for standalone publication in a single part. This article is made available by the Dozenal Society of America, http://www.dozenal.org.