The Dozenal Society of America

Fundamental Operations in the Duodecimal System by Prof. Jay Schiffman

Introduction

A standard assignment in elementary mathematics consists of having the student master addition and multiplication tables where the computations are performed in the decimal system of numeration. In this paper, we endeavor to demonstrate how one performs addition, subtraction, multiplication, and division in the duodecimal (dozenal) system with and without resorting to the addition and multiplication tables. In addition, we solve a number of algebraic equations in dozenal. We initiate our discussion by presenting the standard duodecimal addition and multiplication facts via tables where the symbols χ and Ɛ denote the digits equivalent to decimal ten and eleven.

Before proceeding, a few preliminary remarks are in order. The dozenal system of numeration utilizes one dozen symbols (numerals):

0 1 2 3 4 5 6 7 8 9 χ Ɛ

Figure 1. The dozenal numerals.

MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
10^6	10^5	10^4	10^3	10^2	10^1	10^0
0 ,	0	0	8 ,	1	9	2 .

Table 1. Decimal positional notation.

In decimal positional notation, looking at Table 1 and proceeding from right to left, we have units, tens, hundreds, thousands, etc. We can go on further to ten thousands, hundred thousands, etc. Each place represents a power of ten, again from right to left 10^0, 10^1, 10^2, 10^3 … Thus the example, which is two to the thirteenth power, or the decimal number 8,192, we have eight thousand one hundred ninety two.

"GRAND GROSSES"	GROSS GREAT-GROSSES	DOZEN GREAT-GROSSES	GREAT-GROSSES	GROSSES	DOZENS	UNITS
12^6	12^5	12^4	12^3	12^2	12^1	12^0
0 ,	0	0	4 ,	8	χ	8 ;

Table 2. Dozenal positional notation.

In the dozenal positional notation, each position represents a power of the dozen. Referring to Table 2 and proceeding right to left, we have 12^0, 12^1, 12^2, 12^3… (expressed here and in the table in decimal figures). The positions have names as in decimal: units, dozens, grosses, great grosses, etc. We can proceed further, though the English language does not furnish standard names for powers of the dozen greater than 12^3. Here we furnish "grand grosses" for 12^6 to complete the illustration. Thus, the same number, two to the one dozen first power, is four great gross eight gross ten dozen eight in dozenal. Alternatively, one might say four dozen eight gross ten dozen eight.

At first there appear to be numerous errors in Tables 3 and 4 which non-dozenalists generally interpret in terms of decimal. Let us check some of the number facts. Consider the following dozenal addition problem, and its decimal equivalent below:

$$\chi + Ɛ = 19;$$
$$10 + 11 = 21$$

Now decimal 21 equals dozenal 19;, meaning one group of twelve and nine units, or simply one dozen nine. Similarly,

$$Ɛ + Ɛ = 1\chi;$$
$$11 + 11 = 22$$

that is, one dozen ten.

0	1	2	3	4	5	6	7	8	9	χ	Ɛ
1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	3	4	5	6	7	8	9	χ	Ɛ	10	11
3	4	5	6	7	8	9	χ	Ɛ	10	11	12
4	5	6	7	8	9	χ	Ɛ	10	11	12	13
5	6	7	8	9	χ	Ɛ	10	11	12	13	14
6	7	8	9	χ	Ɛ	10	11	12	13	14	15
7	8	9	χ	Ɛ	10	11	12	13	14	15	16
8	9	χ	Ɛ	10	11	12	13	14	15	16	17
9	χ	Ɛ	10	11	12	13	14	15	16	17	18
χ	Ɛ	10	11	12	13	14	15	16	17	18	19
Ɛ	10	11	12	13	14	15	16	17	18	19	1χ

Table 3 ≈ The Dozenal Addition Table.

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Table 4 ≈ The Dozenal Multiplication Table.

If one next considers multiplication in the usual sense as repeated addition, i.e.,

$$a \times b = (b + b + b + \dots + b),$$
or a addends of b,

then multiplication problems are performed with relative ease. For example, consider the dozenal example 7×8

$$7 \times 8 = (8 + 8 + 8 + 8 + 8 + 8 + 8),$$
or 7 addends of 8,

Let us convert the problem to the decimal system, obtain a decimal product, then convert the product to dozenal:

$$7 \times 8 = 56 \Rightarrow 48;$$

Fifty-six is four groups of twelve and eight units or simply four dozen eight. Refer to Table 4 and observe that decimal fifty-six ($5 \times$ ten $+ 6$) is four dozen eight ($4 \times$ dozen $+ 8$) in duodecimals. Note:

DECIMAL		DOZENAL
$(8+8) = 16$	$\Rightarrow$ 14;	one dozen four
$(8+8+8) = 24$	$\Rightarrow$ 20;	two dozen
$(8+8+8+8) = 32$	$\Rightarrow$ 28;	two dozen eight
$(8+8+8+8+8) = 40$	$\Rightarrow$ 34;	three dozen four
$(8+8+8+8+8+8) = 48$	$\Rightarrow$ 40;	four dozen
$(8+8+8+8+8+8+8) = 56$	$\Rightarrow$ 48;	four dozen eight

Similarly, in dozenal we compute

$$↋ \times ↊ = (↊ + ↊ + ↊ + ↊ + ↊ + ↊ + ↊ + ↊ + ↊ + ↊ + ↊),$$
or ↋ addends of ↊,

Let's again convert the problem to the decimal system, obtain a decimal product, then convert the product to dozenal:

$$11 \times 10 = 110 \Rightarrow 92;$$

One hundred ten is nine groups of twelve and two units or simply nine dozen two. Refer to Table 2 and observe that decimal one hundred ten ($1 \times$ hundred $+ 1 \times$ ten $+ 0$) is nine dozen two ($9 \times$ dozen $+ 2$) in duodecimals.

DECIMAL		DOZENAL
$(10 + 10) = 20$	$\Rightarrow$ 18;	one dozen eight
$(10 + 10 + 10) = 30$	$\Rightarrow$ 26;	two dozen six
$(10 + 10 + 10 + 10) = 40$	$\Rightarrow$ 34;	three dozen four
$(10 + 10 + 10 + 10 + 10) = 50$	$\Rightarrow$ 42;	four dozen two
$(10 + 10 + 10 + 10 + 10 + 10) = 60$	$\Rightarrow$ 50;	five dozen
$(10 + 10 + 10 + 10 + 10 + 10 + 10) = 70$	$\Rightarrow$ 5↊;	five dozen ten
$(10 + 10 + 10 + 10 + 10 + 10 + 10 + 10) = 80$	$\Rightarrow$ 68;	six dozen eight
$(10 + 10 + 10 + 10 + 10 + 10 + 10 + 10 + 10) = 90$	$\Rightarrow$ 76;	seven dozen six
$(10+10+10+10+10+10+10+10+10+10) = 100$	$\Rightarrow$ 84;	eight dozen four
$(10+10+10+10+10+10+10+10+10+10+10) = 110$	$\Rightarrow$ 92;	nine dozen two

The reader is invited to verify other entries in the table.

At this juncture, one is now ready to discuss the arithmetic operations in the duodecimal system. All of our operations will be illustrated by examples.

Dozenal Addition

Example 1. Add 467; + 238;.

STEP 1	STEP 2	STEP 3
1	1	1
467;	467;	467;
+ 238;	+ 238;	+ 238;
3;	↊3;	6↊3;

Solution: Note that decimally $(7 + 8) = 15$, that is, one dozen three or dozenally, 13;. We place the 3 in the units column (the rightmost column) and carry the 1 over to the dozens column (second from the right) — see Step 1 above. In the dozens column, decimally we have $(1 + 6 + 3) = 10$, that is dozenal ↊;. We place the ↊; in the second column. There is no carrying over to the leftmost column, representing multiples of 12^2 (i.e. 144 or one gross) as ↊;= ten is less than the base, which is one dozen — see Step 2 above. Now $(4 + 2) = 6$ in the leftmost (grosses) column — see Step 3 above. Hence, our sum consists of 6 groups of 144, 10 groups of 12, and 3 units, in other words, six gross ten dozen three.

Alternatively, one could convert each addend in the sum to the decimal system of numeration, add using base ten, and then convert the sum to the duodecimal system:

$$\begin{aligned} 467; &= (4 \times 12^2) + (6 \times 12) + (7 \times 1) = 576 + 72 + 7 = 655 \\ +238; &= (2 \times 12^2) + (3 \times 12) + (8 \times 1) = 288 + 36 + 8 = +332 \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 987 \end{aligned}$$

Now convert 987 to a duodecimal number.

Let us note that the place values in base 12 are $\dots 12^3, 12^2, 12, 1$, or $\dots$ 1728, 144, 12, 1. The highest power of the base less than 987 is 12^2 or 144. Successive divisions by the respective powers of the base 12 yields

$$144\overline{)987} = 6,\ 987 - 864 = 123 \qquad 12\overline{)123} = 10 \Rightarrow ↊;,\ 123 - 12 = 3,\ 3 - 0 = 3 \qquad 1\overline{)3} = 3,\ 3 - 3 = 0$$

Hence decimal 987 equals duodecimal 6↊3;.

One can utilize an alternative method for changing a number in the decimal system to the duodecimal system. We illustrate by converting decimal 987 to dozenal. Dividing 987 by 12 yields a quotient of 82 and a remainder of 3. Write the quotient above the dividend and the remainder on the right, as illustrated below, continuing the process of division until the dividend is zero:

STEP 1	STEP 2	STEP 3
		0
	6	12) 6 R6
82	12) 82 R↊	12) 82 R↊
12)987 R3	12)987 R3	12)987 R3

We read the remainders in order from top to bottom, thus decimal 987 equals duodecimal 6↊3;.

Example 2. Add 470; + 947;.

470;	470;	470;
+ 947;	+ 947;	+ 947;
7;	↋7;	11↋7;

Solution: Note that $(0 + 7) = 7$, which is less than the base (12). We note 7 in the units (the rightmost) column. Note that decimally $(7 + 4) = 11 \Rightarrow$ ↋;, which is also less than the base (12). We note ↋ in the dozens (the middle) column. In the leftmost (grosses) column, $(4 + 9) = 13$ decimally, which is one dozen one or dozenally, 11;. We write a one in the grosses column and carry the other one over to the great-grosses ($12^3 = 1728$, decimally) column.

One may resort to the ideas provided in our first example to obtain an alternative solution.

Dozenal Subtraction

Example 1. Subtract 463; − 13↊;.

STEP 1	STEP 2	STEP 3
1	1	1
463;	463;	463;
− 13↊;	− 13↊;	− 13↊;
5;	25;	325;

Solution: Since χ is greater than 3, we must borrow one dozen from the preceding (dozens) column. This yields a sum of (12 + 3) = 15 in decimal notation in the units column. Now subtracting χ ⇒ 10 from 13; ⇒ 15 gives a difference of 5. Complete the problem in the standard manner. The 6 in the dozens column becomes a 5 due to the borrowing; (5 − 3) = 2 in the dozens column, then (4 − 1) = 3. Hence, we have 3 groups of 144, 2 groups of 12, and 5 units, in other words, three gross two dozen five.

We can check our problem by addition:

step 1	step 2	step 3
1	1	1
325;	325;	325;
+ 13χ;	+ 13χ;	+ 13χ;
3;	63;	463;

Note: Decimally (5 + 10) = 15 ⇒ 13;. Place the 3 in the units column and carry the 1 over to the dozens column. In the dozens column, we have (1 + 2 + 3) = 6, which is less than the base, hence we have no further carrying. Next, (3 + 1) = 4, completing the problem.

Example 2. Subtract 1453; − 245;.

1	1	1
1453;	1453;	1453;
− 245;	− 245;	− 245;
χ;	0χ;	120χ;

Solution: Since 5 is greater than 3, we must borrow one dozen from the dozens column. This yields a sum of (12 + 3) = 15 in decimal notation. Now subtracting 5 from 13; ⇒ 15 gives a difference of 10 ⇒ χ. Complete the problem in the usual manner, with no further borrowing required thereafter. The 5 in the dozens column becomes a 4 due to the borrowing; (4 − 4) = 0 in the dozens column. Next (4 − 2) = 2 in the grosses column. Bring down the 1 in the great-grosses column. Hence, we have 1 groups of 1728, 2 groups of 144, no groups of 12, and ten units, in other words, one great gross, two gross five, or one dozen two gross five.

Check by addition:

1	1	1
120χ;	120χ;	120χ;
+ 245;	+ 245;	+ 245;
3;	53;	1453;

Dozenal Multiplication

Multiplication is a bit more involved than addition and subtraction. We illustrate via the following examples:

Example 1. Multiply 124; × 6;.

step 1	step 2	step 3
2	1 2	1 3
124;	124;	124;
× 6;	× 6;	× 6;
0;	20;	720;

Solution: Multiply as one would in decimal, but use the duodecimal multiplication table (Table 4). Multiplying (4 × 6) = 24 ⇒ 20;, or two dozen. We record the products dozenally. Record the 0 in the units (rightmost) column and carry the 2 over to the dozens column (see Step 1 immediately above). In the dozens column, multiply (2 × 6) = 12 ⇒ 10;, adding the two carried from the last operation to get 14 ⇒ 12; or one dozen two. Record the 2 in the dozens column, and carry the 1 over to the grosses (leftmost) column. In the grosses column, multiply (1 × 6) = 6, adding the one carried from the operation in the dozens column to get 7 , which is less than the base (twelve), so no further carrying is necessary. Record the 7 in the grosses column. Hence, we have 7 groups of 144, 2 groups of 12, and no units, in other words, seven gross two dozen; our problem is completed. To check this problem, one could perform the calculation in decimal, then convert the decimal product [1,032] to dozenal notation.

Example 2. Multiply 6χ3; × 24;.

	step 1	step 2	step 3
Phase 1:	1	3 1	3 1
	6χ3;	6χ3;	6χ3;
	× 24;	× 24;	× 24;
	0;	50;	2350;

	step 1	step 2	step 3
Phase 2:		1	1
	3 1	3 1	3 1
	6χ3;	6χ3;	6χ3;
	× 24;	× 24;	× 24;
	2350;	2350;	2350;
	+ 60;	+ 860;	+11860;

	step 1	step 2	step 3
Phase 3:	1	1	1
	3 1	3 1	3 1
	6χ3;	6χ3;	6χ3;
	× 24;	× 24;	× 24;
	2350;	2350;	2350;
	+1186	+1186	+1186
	Ɛ0;	ƐƐ0;	13ƐƐ0;

We've broken the problem into several "phases" merely to fully describe the work. The problem is analogous to the decimal operation, wherein we first multiply the multiplicand (in the example, 6χ3;) by the smallest digit of the multiplier (in the example the "4" in 24;), then proceed to multiply the multiplicand by the next digit in the multiplier, etc., each time generating a partial product for each digit of the multiplier. Once all multiplication by the digits of the multiplier is complete, we total all the partial products we've obtained through multiplication.

Solution: In Phase 1, we are simply multiplying by the rightmost digit of the multiplier 24;, i.e. 4 units, to obtain the first of two partial products. Multiplying (4 × 3) = 12 ⇒ 10;, or one dozen. We record the products dozenally. Record the 0 in the units (rightmost) column and carry the 1 over to the dozens column (see Phase 1 Step 1 immediately above). In the dozens column, multiplying (4 × 10) = 40 ⇒ 34;, or three dozen four, then adding the 1 carried over from the last operation, we have 35; or three dozen five. Write the 5 in the dozens column, carry the 3 to the grosses column (Phase 1 Step 2). Next, we multiply the grosses position of the multiplicand, (4 × 6) = 24 ⇒ 20;, or two dozen, then adding the 3 carried over from the dozens operation, we have 23; or two dozen three. Write the 3 in the grosses column, and carry the 2 to the great-grosses column, as in Phase 1 Step 3.

In Phase 2, we multiply the next digit of the multiplier 24;, i.e. 2 dozens, to get the second partial product. Multiplying (2 × 3) = 6. Record the 6 in the dozens column — just as in decimal multiplication, the products must be noted directly under the position of the multiplier, leaving an empty units column, in this case (see Phase 2 Step 1). Next we multiply (2 × 10) = 20 ⇒ 18;, or one dozen eight. Write the 8 in the grosses column, carry the 1 to the great-grosses column (Phase 2 Step 2). Next, we multiply the grosses position of the multiplicand, (2 × 6) = 12 ⇒ 10;, or one dozen, then adding the 1 carried over from the last operation, we have 11; or one dozen one. Write the 1 in the great-grosses column, and carry the 1 to the dozen-great-grosses column, as in Phase 2 Step 3.

In Phase 3, we add the two partial products. The addition progresses as described previously in the "Dozenal Addition" section. Observe that after bringing down the "0" in the units-column of the first partial product, (5 + 6) = 11 ⇒ Ɛ, which is less than the base (see Phase 3 Step 1). Next, (3 + 8) = 11 ⇒ Ɛ (Phase 3 Step 2), then (2 + 1) = 3 and finally we simply bring down the 1 in the dozen-great-grosses column (Phase 3 Step 3). The problem is complete. Hence, we have one dozen great gross three great gross eleven gross eleven dozen. We might also say one gross three dozen eleven gross eleven dozen. Again, we could check the product by performing the calculation in decimal, then converting the decimal product [27,636] to dozenal notation.

Dozenal Division

Division is the most involved operation encountered so far. It is instructive to illustrate via the following problems:

<u>Example 1</u>. Divide 431; ÷ 6;.

```
STEP 1        STEP 2        STEP 3
   8             86            86 R1
6)431         6)431         6)431
 – 40          – 40          – 40
    3            31            31
               – 30          – 30
                                1
```

Referring to the dozenal multiplication table (Table 4), we extract the multiples of the divisor, 6:

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
6	10	16	20	26	30	36	40	46	50	56	60

Using the topmost row as an index which supplies multipliers and the leftmost column to find 6, the following set of facts are established:

DECIMAL		DOZENAL
$(6 \times 1) = 6$	$\Rightarrow$ 6;	six
$(6 \times 2) = 12$	$\Rightarrow$ 10;	one dozen
$(6 \times 3) = 18$	$\Rightarrow$ 16;	one dozen six
$(6 \times 4) = 24$	$\Rightarrow$ 20;	two dozen
$(6 \times 5) = 30$	$\Rightarrow$ 26;	two dozen six
$(6 \times 6) = 36$	$\Rightarrow$ 30;	three dozen
$(6 \times 7) = 42$	$\Rightarrow$ 36;	three dozen six
$(6 \times 8) = 48$	$\Rightarrow$ 40;	four dozen
$(6 \times 9) = 54$	$\Rightarrow$ 46;	four dozen six
$(6 \times 10) = 60$	$\Rightarrow$ 50;	five dozen
$(6 \times 11) = 66$	$\Rightarrow$ 56;	five dozen six
$(6 \times 12) = 72$	$\Rightarrow$ 60;	six dozen

Solution: Looking at the first digit of the dividend, i.e. the "4" in the grosses place in the number 431;, we see it is less than the divisor, 6. Thus we take the first two digits of the dividend, i.e. "43". Since (6 × 8) = 48 ⇒ 40; (four dozen), which is less than 43; (four dozen three), 6 goes into 43; eight times. Write an 8 as a trial digit of the quotient above the smallest digit of the portion of the dividend under consideration (i.e. the "3" in "43"), then write the product of the divisor and the trial digit of the quotient, "40", under "43". Subtract 40; from 43; to get 3;. (See Step 1 immediately above). Grab the next digit from the dividend (i.e. "1" from "431") and place it to the right of the "3" obtained from the last operation. Examining the excerpted multiplication facts for 6 ×, we see that (6 × 6) = 36 ⇒ 30; (three dozen), which is less than 31; (three dozen one), 6 goes into 31; six times. Write the "6" as the next digit of the quotient and subtract 30; from 31; to obtain 1. (See Step 2.)

If we are not interested in a fractional, we can now note the 1 left over from Step 2 as a remainder (see Step 3). Hence, the quotient of 431; ÷ 6; is 86; (eight dozen six) with a remainder of 1.

```
STEP 3        STEP 3-ALT
   86 R1          86;2
6)431         6)431;0
 – 40          – 40
   31            31
 – 30          – 30
    1             1;0
                – 1;0
                   ;0
```

If we need more digits in the quotient, say, if we wanted to compute the quotient to three significant digits, we could grab a zero (as the next digit of the dividend 431; after all the digits were exhausted, would be "0") and place it to the right of the 1 obtained in step 3. Examining the excerpted multiplication facts for 6 ×, we see that (6 × 2) = 12 ⇒ 10; (one dozen), which is equal to 10;, 6 goes into 10; twice. Write the "2" as the next digit of the quotient, after the dozenal unit point (;) and subtract 10; from 10; to obtain 0. Carrying the problem out to three places, coincidentally, we end up with no remainder. (See Step 3-alt.) For many other division problems, a remainder is still generated, and one may need to compute one further digit as described in Step 3-alt, i.e. by grabbing another zero from the dividend, to obtain the next digit of the quotient. If this next digit is greater than half the base (i.e. 6), then one rounds up the preceding digit. If the next digit is less than half the base, one preserves the preceding digit. This is just as is the usual decimal operation, only rounding above 6 rather than 5. Note that the remainder 1 when the divisor is 6 is ⅙ which is equal to 0;2 in duodecimal. Another way to express the quotient is as a mixed number: 86;⅙. Thus Step 3 and Step 3-alt, in cases where the remainder of Step-3-alt is zero, are equivalent in all but notation.

We can check the problem by observing that (*quotient* × *divisor*) + *remainder* = *dividend*. Thus, (86; × 6;) + R1 should equal 431;:

```
CHECK                    CHECK-ALT
   86;                      86;2
×   6;                   ×    6;
  430; + R1 = 431;         431;
```

<u>Example 2</u>. Divide 235; ÷ 7;.

```
STEP 1        STEP 2        STEP 3
   3             3Ɛ            3Ɛ
7)235         7)235         7)235
 – 19          – 19          – 19
    6            65            65
               – 65          – 65
                                0
```

Referring to the dozenal multiplication table (Table 4), we extract the multiples of the divisor, 7:

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70

Solution: Examing the first digit of the dividend, i.e. the "2" in the grosses place in the number 235;, we see it is less than the divisor, 7. Thus we take the first two digits of the dividend, i.e. "23". Since (7 × 3) = 21 ⇒ 19; (one dozen nine), which is less than 23; (two dozen three), 7 goes into 23; three times. Write a 3 as a trial digit of the quotient above the smallest digit of the portion of the dividend under consideration (i.e. the "3" in "23"), then write the product of the divisor and the trial digit of the quotient, "19", under "23". Subtract 19; from 23; to get 6;. (See Step 1 immediately above). Grab the next digit from the dividend (i.e. "5" from "235") and place it to the right of the "6" obtained from the last operation. Examining the excerpted multiplication facts for 7 ×, we see that (7 × 11) = 77 ⇒ 65; (six dozen five), which equal to 65;, 7 goes into 65; exactly eleven times. (See Step 2.) Write the "Ɛ" as the next digit of the quotient and subtract 65; from 65; to obtain 0. There is no remainder to this problem. (See Step 3.) Hence, the quotient of 235; ÷ 7; is exactly 3Ɛ; (three dozen eleven).

Checking the problem, we observe that (3Ɛ; × 7;) should equal 235;:

```
CHECK
   3Ɛ;
×   7;
  235;
```

Solving Dozenal Equations

Recall that an equation is a mathematical statement of equality between two expressions. In the decimal system, "2 + 5 = 7" and "3 × 7 = 21" are examples of equations. The equation "2 + 8 = 11" is another example, although it is a false statement.

The following examples are equations (and true statements) in the duodecimal system:

$$Ɛ + ↊ = 19; \qquad ↊ - 9 = 1 \qquad 8 \times Ɛ = 74; \qquad 1↊; \div 2 = Ɛ$$

One can solve dozenal equations by a process analogous to that in the decimal system. We will illustrate through the following examples.

<u>Example 1</u>. Solve for x: $x + ↊ = Ɛ$.

Solution: From our duodecimal addition table (Table 3) we desire the number that precedes Ɛ by ↊ units. It is easily determined that x = 1. Of course one may subtract a units from both sides of the equation as shown below:

$$\begin{array}{rl} x + ↊ = & Ɛ \\ \underline{\;-↊ =} & \underline{-↊} \\ x \quad = & Ɛ - ↊ \\ x \quad = & 1 \end{array}$$

Converting the problem to decimal notation, $x = (Ɛ - ↊) \Rightarrow (11 - 10) = 1$, which is one unit either decimally or dozenally. Thus $x = 1$.

<u>Example 2</u>. Solve for n: $n - 145; = 789;$.

Solution: One can add 145; to both sides:

$$\begin{array}{rl} n - 145; = & 789; \\ \underline{+145; =} & \underline{+145;} \\ n \qquad = & 789; + 145; \end{array}$$

Thus, add:

step 1	*step 2*	*step 3*
1	1 1	1 1
789;	789;	789;
+ 145;	+ 145;	+ 145;
2;	12;	912;

Note that decimally (9 + 5) = 14 ⇒ 12; (one dozen two). Record the 2 in the units column (the rightmost column) and carry the 1 over to the dozens column (see Step 1 above). In the dozens column, decimally we have (1 + 8 + 4) = 13, that is dozenal 11;. We place the 1; in the second column and carry the 1 over to the grosses column (see Step 2 above). Now (1 + 7 + 1) = 9 in the grosses column (see Step 3 above). Hence, our sum consists of 9 groups of 144, 1 groups of 12, and 2 units, in other words, nine gross one dozen two. Hence, n = 912;.

<u>Example 3</u>. Solve for m: $7m = 53;$.

Solution: It is easy to solve this problem using the dozenal multiplication table (Table 4). Observe that 7 goes into 53; precisely 9 times. Hence $m = 9$.

<u>Example 4</u>. Solve for m: $7m = 54;$, carry the quotient out to 7 places.

```
   9;186↊35
 7)54;000000
 – 53
     1;0
   – ;7
      ;50
    –;48
      ;040
      –36
        60
       –5↊
         20
        –19
          30
         –2Ɛ
           1
```

Solution: This problem is related to the one above, save that 7 does not go into 53; precisely; it generates a remainder of 1. We can state the answer to this example as a mixed quotient (9 r1), a mixed number (9¹⁄₇), or an improper fraction (⁵⁴;⁄₇). Carrying out the calculation to seven places involves long division (shown immediately at right). Note that at the end of the process, we still have a remainder. If we were to continue division, we would observe that the sequence "186↊35" recurs. This is because the digit 7 is coprime to the base. (In dozenal, the digits 1, 5, 7, and Ɛ are coprime to the base. In decimal, 1, 3, 7, and 9 are coprime to ten.)

<u>Example 5</u>. Solve for d: $d/6 = 8$.

Solution: Multiply both sides by 6 (refer to the dozenal multiplication table, Table 4):

$$\begin{array}{rl} 6 \cdot (d/6) = & 6 \cdot 8 \\ d = & 40; \end{array}$$

Thus d = 40;, four dozen.

It is easy to solve this problem using the dozenal multiplication table (Table 4). Observe that 7 goes into 53; precisely 9 times. Hence $m = 9$.

<u>Example 6</u>. Solve for p: $2p + 50; = 68;$.

Solution: First, subtract 50; from both sides, then divide both sides by the factor 2:

$$\begin{array}{rl} 2p + 50; = & 68; \\ \underline{-50; =} & \underline{-50;} \\ 2p \qquad = & 68; - 50; \\ 2p \qquad = & 18; \\ p \qquad = & 18; \div 2 \\ p \qquad = & ↊ \end{array}$$

From the dozenal multiplication table, we observe that one dozen eight divided by two is ten. Hence, $p = ↊$ is our solution.

<u>Example 7</u>. Solve for a: $a^2 = ↊1;$.

Solution: From our table for duodecimal multiplication we see that $Ɛ \times Ɛ = Ɛ^2 = ↊1$. Since either a negative or positive value of a, squared, equals a positive number, our solution has two possible answers. Our solution is $a = \{Ɛ, -Ɛ\}$ or $\pm Ɛ$, as both $Ɛ^2$ and $(-Ɛ)^2 = ↊1;$ or ten dozen one.

Conclusion

The duodecimal system of numeration presents a very neat illustration of how to perform basic arithmetic and algebraic computations in bases other than decimal. In this paper, the author has attempted to demonstrate these ideas. Other activities, such as digital fractional computations, can be performed in much the same manner as in the decimal system. The use of another system of numeration serves to provide variety and promotes interest and appeal to mathematicians and dozenalists alike. The reader is invited to seek other activities in which duodecimals can be utilized.

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This document was remastered 26 January 2011 by Michael Thomas De Vlieger. The following alterations to the original were made mainly for consistency with the current (2011.) documentation conventions, exhibited by documents available at www.Dozenal.org and in the *Duodecimal Bulletin* beginning with Vol. 49; №. 2, wn 97;, 11Ɛ4; (2008.) The term "twelve" and the powers of the dozen were replaced by English-language words for the dozen and its powers, i.e. "dozen", "gross", "great-gross", "dozen great-gross", etc. for decimal $12^1, 12^2, 12^3, 12^4$, etc., respectively. A brief explanation of positional notation in decimal and dozenal was added. All dozenal numerals are expressed using the W. A. Dwiggins or dsa-Classic symbology, with digit-ten expressed as ↊ and digit-eleven as Ɛ, whereas the original used the "Bell" transdecimal numerals {✗, ♯}. Except in examples, decimal numbers are formatted using "old style" numerals (12345) and dozenal using plain numerals to align with the above-mentioned conventions. The problems are now formatted so that they display stages of work. Addition Example 2 was changed to generate a carry in the "grosses" column. The multiplication tables were produced in a different order, multiplication consistently following addition in this document. The supplementary decimal-dozenal comparison tables (Tables 5 & 6), originally conceived by Mr. Fred Newhall, have been reformatted to align with the left-to-right, top-to-bottom format of the traditional multiplication and addition tables, rather than the original right-to-left. Tables appear in this document in places that differ from those in the original. The original document employed dot-matrix printing characteristic of the late 1980s. Mr. De Vlieger's objective is a clearer remastered document.

The tables from the article have been reprinted here for easy reference.

MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	THOUSANDS	HUNDREDS	TENS	UNITS
10^6	10^5	10^4	10^3	10^2	10^1	10^0
0 ,	0	0	8 ,	1	9	2 .

Table 1. Decimal positional notation.

"GRAND GROSSES"	GROSS GREAT-GROSSES	DOZEN GREAT-GROSSES	GREAT-GROSSES	GROSSES	DOZENS	UNITS
12^6	12^5	12^4	12^3	12^2	12^1	12^0
0 ,	0	0	4 ,	8	χ	8 ;

Table 2. Dozenal positional notation.

0	1	2	3	4	5	6	7	8	9	χ	Ɛ
1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	3	4	5	6	7	8	9	χ	Ɛ	10	11
3	4	5	6	7	8	9	χ	Ɛ	10	11	12
4	5	6	7	8	9	χ	Ɛ	10	11	12	13
5	6	7	8	9	χ	Ɛ	10	11	12	13	14
6	7	8	9	χ	Ɛ	10	11	12	13	14	15
7	8	9	χ	Ɛ	10	11	12	13	14	15	16
8	9	χ	Ɛ	10	11	12	13	14	15	16	17
9	χ	Ɛ	10	11	12	13	14	15	16	17	18
χ	Ɛ	10	11	12	13	14	15	16	17	18	19
Ɛ	10	11	12	13	14	15	16	17	18	19	1χ

Table 3 ≈ The Dozenal Addition Table.

1	2	3	4	5	6	7	8	9	χ	Ɛ	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2Ɛ	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2Ɛ	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	χ0
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1	Ɛ0
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100

Table 4 ≈ The Dozenal Multiplication Table.

DECIMAL

DOZENAL

		1	2	3	4	5	6	7	8	9	10	11	12
		2	3	4	5	6	7	8	9	10	11	12	13
1	2		4	5	6	7	8	9	10	11	12	13	14
2	3	4		6	7	8	9	10	11	12	13	14	15
3	4	5	6		8	9	10	11	12	13	14	15	16
4	5	6	7	8		10	11	12	13	14	15	16	17
5	6	7	8	9	χ		12	13	14	15	16	17	18
6	7	8	9	χ	Ɛ	10		14	15	16	17	18	19
7	8	9	χ	Ɛ	10	11	12		16	17	18	19	20
8	9	χ	Ɛ	10	11	12	13	14		18	19	20	21
9	χ	Ɛ	10	11	12	13	14	15	16		20	21	22
χ	Ɛ	10	11	12	13	14	15	16	17	18		22	23
Ɛ	10	11	12	13	14	15	16	17	18	19	1χ		
10	11	12	13	14	15	16	17	18	19	1χ	1Ɛ		

Table 5 ≈ Decimal–Dozenal Addition Comparison Table.

DECIMAL

DOZENAL

Squares appear in red along this diagonal border.

		1	2	3	4	5	6	7	8	9	10	11	12
			4	6	8	10	12	14	16	18	20	22	24
1				9	12	15	18	21	24	27	30	33	36
2	4				16	20	24	28	32	36	40	44	48
3	6	9				25	30	35	40	45	50	55	60
4	8	10	14				36	42	48	54	60	66	72
5	χ	13	18	21				49	56	63	70	77	84
6	10	16	20	26	30				64	72	80	88	96
7	12	19	24	2Ɛ	36	41				81	90	99	108
8	14	20	28	34	40	48	54				100	110	120
9	16	23	30	39	46	53	60	69				121	132
χ	18	26	34	42	50	5χ	68	76	84				144
Ɛ	1χ	29	38	47	56	65	74	83	92	χ1			
10	20	30	40	50	60	70	80	90	χ0	Ɛ0	100		

Table 6 ≈ Decimal–Dozenal Multiplication Comparison Table.

Tables 5 and 6 appeared in similar form on pages 9 and ♮ (Ɛ) of Vol. 32; №. 2,
wn 62;, pages 4–11;, 1199; (1989.) Fred Newhall supplied the original charts.
The charts combine both decimal and dozenal multiplication tables.