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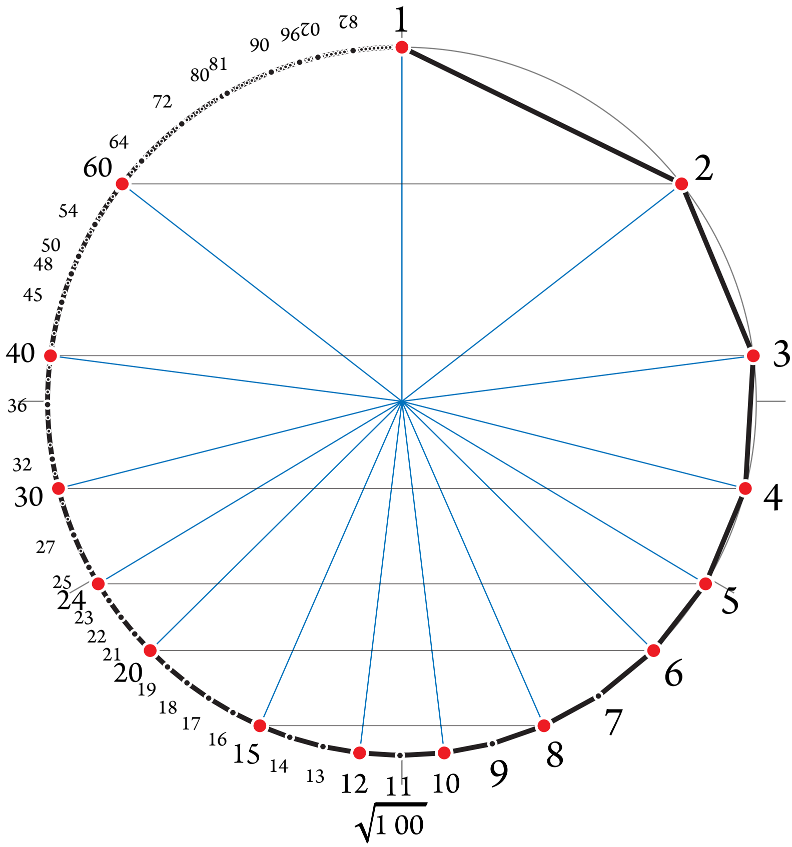

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The Dozenal Society of America

is a voluntary nonprofit educational corporation, organized for the conduct of research and education of the public in the use of base twelve in calculations, mathematics, weights and measures, and other branches of pure and applied science

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The Duodecimal Bulletin

Celebrating Ten Dozen Issues!

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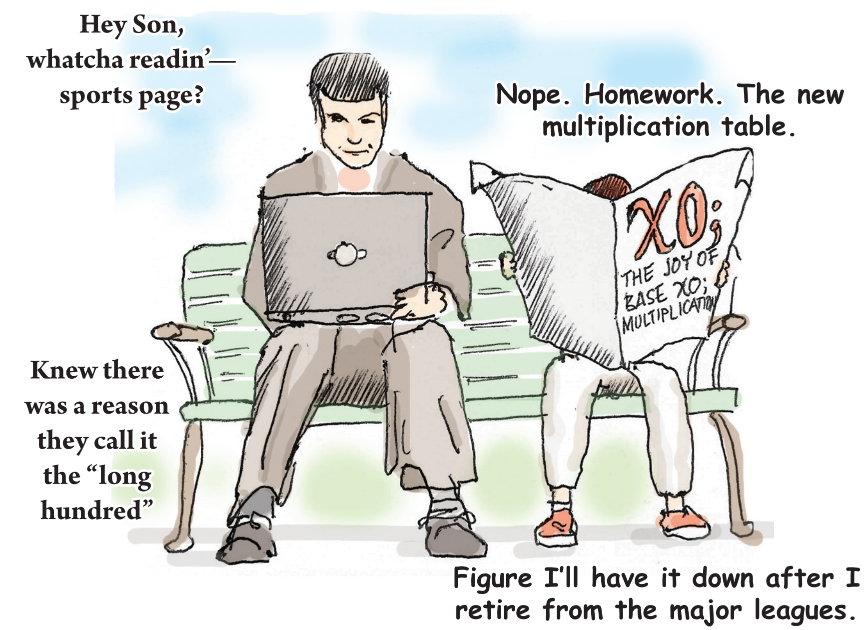


president's message

Welcome to a new "duodecade" of your *Duodecimal Bulletin*! In honor of this ten dozenth issue of the *Bulletin*, we'll examine the number dek-do and other highly composite numbers of its ilk. Our Germanic forefathers knew ten dozen as the "great" or "long hundred", we'll read about our Society's encounter with this concept in a couple reprints of our accounts of Danish native Jens Ulff-Møller's visit with us in the eighties. Australian Wendy Krieger contributes a few items on the number she affectionately calls "twelfty". Bill Lauritzen, a Californian teaching in southern China, introduces us to the concept of "versatile numbers". Finally, we'll examine the dozenal division of a circle and compare it to the decimal 360-degree system.

The last mentioned article, on measurement and number base, introduces a department we'll introduce across some of the next dozen issues looking into various systems of dozenal measure. We're planning to examine Tom Pendlebury's TGM, celebrated in the UK, the *Manual of the Dozen System* introduced by this Society in 1960, and Takashi Suga's Universal Unit System. For this to be successful, we'd like your help. Do you have a favorite system? Why do you espouse that system? What qualities appeal to you in a system of weights and measure? We're eager to hear what you have to say.

The Dozenal Society of America is always looking for the participation of its Members, because part of being a Society is being social. A few of the Members met in December 2010 in New York for dinner and conversation. Consider joining us in June at Nassau Community College for our Annual Meeting. There are plenty of dozenal doings to partake in across the year. Some of you may be particularly motivated to contribute articles to this *Bulletin*, or your own ideas to our website. All our work is volunteered, so we certainly appreciate any contribution you can make. If you think something could be done better, please lend a hand. We appreciate your Membership, because your participation and your dues help keep this Society moving forward.

I hope you enjoy this ten dozenth issue of the *Duodecimal Bulletin*. We've tapped contributors far and wide for their thoughts; I do hope these stimulate your own thoughts. The Society looks forward to seeing you in New York City this summer!

The Dozen in Daily Life

11Ɛ7; · Minutes of the Annual Meeting · 2011.

21; (25.) June 11Ɛ7; (2011.)

DSA Board Meeting was called to order at about 2:00 PM.

The Board noted that long-time DSA stalwart Alice Berridge died at Ɛ;00 this very morning. She was active in the DSA from day one. Over the years Alice served as Vice President, Secretary, and Treasurer of the Society. She also worked on many committees: Nominating, Meeting, and Awards to name a few. She educated people about dozenal counting and promoted the DSA tirelessly among her family and friends. Her presence will be sorely missed by one and all.

The Treasurer, Jay Schiffman, reported positive growth in both DSA membership and in DSA investments from 2010–2011.

The Outreach and Education Committee will consist of Secretary Jen Seron, plus volunteers from membership.

The slate of officers proposed by the Nominating Committee consisted of the following: Board Chair Jay Schiffman, President Michael De Vlieger, Vice-President Graham Steele, Treasurer Jay Schiffman, Secretary Jen Seron. There being no other nominations, the Chair was directed to cast one vote for the slate. Michael De Vlieger was reappointed Editor of our Bulletin. Gene Zirkel was appointed Parliamentarian to the Chair.

The Board Meeting was adjourned unanimously at 3:45 PM.

Annual Meeting of the Dozenal Society of America was called to order around 3:15 PM

At the general membership meeting:

The DSA's incorporation documents need to updated to align with NY State non-profit 501C law. President Michael De Vlieger reported on the amendment of corporate documents to accommodate changes in NY State incorporation law. These amendments will put us in compliance with that law. This amendment was approved unanimously.

President De Vlieger reported on the traffic of the recently revised and updated DSA website. Old copies of the Bulletin are now available online.

Vice-President Graham Steele has set up a Facebook site, and President De Vlieger has set up a Twitter account.

The Board of Directors Class of 2014. was elected as follows:

Michael DeVlieger
Jen Seron
Brian Ditter
Donald Goodman

A lively discussion of dozenal ideas was led by Brian and Mike. All joined in. Then the members retired to a nearby restaurant for supper and more discussion.

MOVE OVER PRIMES—

Versatiles Are Here!

by Bill Lauritzen

Although we teach students what prime numbers are, there is another class of numbers that is just as important as prime numbers that we do not teach students. I call these versatile numbers.

A versatile number is a number that has more factors than any smaller number. The first versatile numbers are:

2, 4, 6, 10, 20, 30, 40, 50, χ0, 130, 180, 260, 500, 5χ0, 890, Ɛ80, 1560, 2Ɛ00…

Figure 1: The Highly Composite Numbers (hcn), Sloane's oeis: a002182.

They facilitate the even sharing of things.

The British mathematician, Hardy, called these numbers, "as unlike a prime as a number can be." However, like primes they are infinite, and like primes they cannot be predicted by any rule.

At the dawn of civilization the Babylonians chose one these numbers for their base 50; numeration. They also used some of these numbers to divide time and the circle of the heavens. These are still in use today. I'm told by scholar Meyer Rainer, that Plato related some of these numbers to the five regular ("Platonic") polyhedra and chose one of these numbers, 2Ɛ00;, with which to populate his ideal city-state. In the last century, (1915. to be more precise), the great Indian mathematician Ramanujan, identified these numbers as "highly composite numbers." He calculated them to the eight dozen sixth in the sequence, which is 90Ɛ,597,889,000. He also proved several interesting things about these numbers. However, despite this attention, these numbers have remained relatively hidden from the population, and research into them has been considered off the mainstream of mathematics.

Sharing 10; Objects

Kids	Objects	Objects per Kid
★ 1		10
★ 2		6
★ 3		4
★ 4		3
5		2;49724..
★ 6		2
7		1;86χ351..
8		1;6
9		1;4
χ		1;24972..
Ɛ		1;111111..
★ 10		1

Figure 1: Sharing a dozen objects.

Unaware of most of this, in 1995., I rediscovered these numbers and named them "antiprimes" or "versatile numbers." I discovered that these numbers had the property that THEY FACILITATE THE EVEN SHARING OF THINGS. Because of this I hypothesized that IGNORANCE OF THIS CLASS OF NUMBERS HAS INCREASED THE AMOUNT OF VIOLENCE IN THE WORLD. Later, I realized that cryptography (secret codes) uses primes to keep things hidden. In a like manner, perhaps society could use versatile numbers to keep things open. (Before teaching mathematics I was a psychologist so this cross-pollination of thought is not to be unexpected.

For a more detailed discussion of these numbers, visit Prof. Lauritzen's website at www.earth360.com

I proposed an experiment in which small groups of children (of from 2 to 6 individuals, as these are the most commonly seen sizes) could be put in a room and then given either a versatile 10; or a non-versatile Ɛ pieces of candy to share. A specially trained observer, who knew nothing of the purpose of the experiment, would count incidences of hostile aggression among the children. I predict that with 5 children there would be no difference in the behavior of the children with versatile as opposed to non-versatile pieces of candy. However, with 2, 3, 4, or 6 children I predict that there will be more incidences of hostile aggressive behavior with the non-versatile Ɛ pieces of candy than with the versatile 10; pieces of candy. Try the experiment yourself and draw your own conclusions.

Sharing Ɛ Objects

Kids	Objects	Objects per Kid
★ 1		Ɛ
2		5;6
3		3;8
4		2;9
5		2;249724..
6		1;χ
7		1;6χ3518..
8		1;46
9		1;28
χ		1;124972..
★ Ɛ		1

Figure 2: Sharing eleven objects.

(Editor's Note: See Figure 1 for 10; and Figure 2 for Ɛ students. In the figures, the number of students in the "Kids" column divide the number of objects stated in the title of the chart. The "Objects" column visually displays the number of objects each child receives if the total number of objects are divided equally among each child. Any fractional object is represented by a red "explosion". The "Objects per Kid" column gives the dozenal equivalent of the total number of objects divided by the number of equally-sharing kids shown in the leftmost column of the same line. Stars at the leftmost column indicate an integral number of objects can be evenly distributed among the children.)

The experiment could be repeated with a versatile 10; versus a non-versatile χ pieces of candy. (See Figure 3). In this case I predict a) no difference with 2 children, b) less hostile aggression with 5 children and χ pieces of candy, and c) less hostile aggression with 3, 4, and 6 children with 10; pieces of candy. In other words, there are more opportunities for hostile aggression with a non-versatile ten than with a versatile dozen. Again, I leave it to you to do the experiment and then draw your own conclusions about the value of the metric system and our decimal number system versus a more versatile measuring system and a more versatile numbering system.

Regardless of your conclusions in these controversial areas, I think you will find in your daily life many examples where versatile numbers are being used intuitively by yourself and others already. For example, a teacher with a versatile 20; students in the classroom can easily divide the students into even groups of 2, 3, 4, 6, 8, or 10; students. A teacher with a non-versatile 1Ɛ; students has no such options. A versatile 20; acre lot can easily and evenly divided up, but with a 21; acre lot it is not as easy. Merchants use versatile six-packs, versatile dozens, versatile 10;-20;-30; pictures in a roll of film, and so on.

Sharing χ Objects

Kids	Objects	Objects per Kid
★ 1		χ
★ 2		5
3		3;4
4		2;6
★ 5		2
6		1;8
7		1;5186χ3..
8		1;3
9		1;14
★ χ		1

Figure 3: Sharing ten objects.

If you study versatile numbers, I think you will find even more areas in both your personal life and in teaching in which you can use these numbers. I think you will find that students that are taught these numbers can better apply mathematics. In summary, I think you will conclude that knowing versatile numbers is of value.

№	Factors	σ
1	1 × 1	1
★2	1 × 2	2
3	1 × 3	2
★4	1 × 4, 2 × 2	3
5	1 × 5	2
★6	1 × 6, 2 × 3	4
7	1 × 7	2
8	1 × 8, 2 × 4	4
9	1 × 9, 3 × 3	3
χ	1 × χ, 2 × 5	4
Ɛ	1 × Ɛ	2
★10	1 × 10, 2 × 6, 3 × 4	6
11	1 × 11	2
12	1 × 12, 2 × 7	4
13	1 × 13, 3 × 5	4
14	1 × 14, 2 × 8, 4 × 4	5
15	1 × 15	2
16	1 × 16, 2 × 9, 3 × 6	6
17	1 × 17	2
18	1 × 18, 2 × χ, 4 × 5	6
19	1 × 19, 3 × 7	4
1χ	1 × 1χ, 2 × Ɛ	4
1Ɛ	1 × 1Ɛ	2
★20	1 × 20, 2 × 10, 3 × 8, 4 × 6	8
21	1 × 21, 5 × 5	4
22	1 × 22, 2 × 11	4
23	1 × 23, 3 × 9	4
24	1 × 24, 2 × 12, 4 × 7	6
25	1 × 25	2
26	1 × 26, 2 × 13, 3 × χ, 5 × 6	8
27	1 × 27	2
28	1 × 28, 2 × 14, 4 × 8	6
29	1 × 29, 3 × Ɛ	4
2χ	1 × 2χ, 2 × 15	4
2Ɛ	1 × 2Ɛ, 5 × 7	4
★30	1×30, 2×16, 3×10, 4×9, 6×6	χ

Figure 4.

When I posted information on versatile numbers to the "sci.math" newsgroup on the Internet, I received more e-mail than I could easily keep up with. In St. Petersberg, Russia, teacher Roman Breslav held a Versatile Number Day in the Math Center of the Palace of Youth Creativity.

There may be better ways to teach versatile numbers than the following, but so far this is the best method I have discovered. Simply have the students make a factor table like Figure 4.

I find that it works well to let a different student come to the board and fill in a row. When we get to 6 rows we erase it and start on the next 6 rows. Whenever a number has only 2 factors it is of course a prime and could be circled. Whenever the "number of factors" (marked by "σ" in the chart for brevity) goes higher than all the previous numbers of factors, a star (★) could be placed next to the number to mark a versatile number. Students can make this table, depending on their grade level, to two dozen or higher. I have had many middle school students make it up to five dozen. High school students could make it up to 120, or, if they are advanced enough, they can write a simple computer program that will go much higher.

As you are making this table with the students you can ask them questions like:

1) Which has more factors? χ or 10;? (You may be surprised to find how many students say χ even though the evidence is right in front of their noses.)

2) Which has more factors? 10; or 13;?

3) Which has more factors? 20; or 21;?

4) Why do you think we have 10; hours on the clock instead of χ?

5) Why do you think we have 20; hours in a day instead of 21;?

6) Why do you think we have 50; minutes in a clock instead of 42;?

In summary, I think the use of these numbers can lubricate social interaction in an increasingly populated and tense world. In my opinion, widespread knowledge of these numbers may never before have existed in the history of human civilization. Whether these numbers will remain hidden from the general public, or will become a part of our standard curriculum, remains largely in your hands.

History Professor Speaks to DSA Members

Reprint (Originally published in Vol. 34; №. 2 wn 68; pages 16;–17;)

On Tuesday, 12; May 119Ɛ;, on very short notice, history professor Jens Ulff-Møller from Copenhagen conducted a brief but interesting seminar at Nassau Community College on the history of counting and measuring. He was returning home to Denmark from the 26th Annual Congress on Medieval Studies held at Western Michigan University in Kalamazoo, where he spoke and also organized two sessions of a half dozen speakers. We were indeed fortunate to secure his presence as he spoke about a variety of topics relating to medieval counting and measurement.

It was very interesting to learn of the difficulties historians and linguists had in deciphering such phrases as *'a year had three hundred and five days'*. The problem is not that our ancestors used shorter years, but rather they used longer 'hundreds'.

According to Professor Ulff-Møller, many people used a hundred containing dek dozen units. We refer to this today as the long hundred, and differentiate it from the narrow hundred of only eight dozen and four units. (You may recall the long ton of 2240* — rather than 2000 — pounds, still in use today.)

In the middle ages, most numbers were written out, and algorithms for operations were not easy to come by, division being an especially vexing problem. This led people to desire that the number of partitions in a given unit of measurement be highly factorable and hence — the long hundred of dek dozen units. It divides evenly by {1, 2, 3, 4, 5, 6, 8, χ, 10, 13, 18, 20, 26, 34, 50, χ0}.

Plato's perfect world may have led academicians to prefer the regularity of either dek times dek or do times do as the number of subdivisions of a measurement, but the common people did things for convenience and the result was the long hundred.

This preference for convenience over standardization may be the reason we had so many hybrid combinations of partitions of our units of measurement. (These were to be found on the back cover of our black and white notebooks in grammar school.) A preference for convenience may also explain the present resistance to being forced to adopt the awkward decimal metric system. People everywhere seem to demand halves and then thirds and/or quarters in their measurements in order to avoid fractions of units as much as possible.

We tend to think of things being codified and universal. However many measurements were regional and things were written differently in different localities. Thus we find Roman numerals not quite as standard as we were taught in elementary school. For example:

IV is not the only four — IIII was also used. Two hundred appears as II hundred as well as CC. But sometimes CC stands for two long hundreds! V^{XX} is found as denoting 5 times 20 and VI is used for six thousand.

The reader who is not aware of these variations would have difficulty in attempting to decipher the meaning of some passages in medieval texts.

To some extent, it appears that the popular culture preferred the long hundred while the narrow hundred prevailed in sacral use.

Professor Ulff-Møller is due to return to the States in November. We hope to be able to announce a date when he will speak to us again, and we trust that you will have the opportunity to hear him. Don't be surprised if you read that:

100 fish = 6 score fish, 3 (units) of fish = 100, and 1(unit) = 40.

* *In the original article at 34216;, Prof. Zirkel wrote 2400 rather than 2240 pounds to the long ton.*

A Dozen Decades

A Report on a Lecture About Long Hundreds and Historical Methods

Reprint (Originally published in Vol. 35; №. 1 wn 6Ӿ; pages 11;–12;)

All numbers are decimal in this article.

On Thursday [7 November 1991], the Dozenal Society of America, in conjunction with the departments of Mathematics/Computer Processing/Statistics, and of History at Nassau Community College (LI) sponsored a lecture by Jens Ulff-Møller, PhD candidate from the University of Copenhagen. Last year, Jens, on very short notice, gave a brief presentation which touched upon some of the problems translators face when trying to interpret numerical data in medieval documents, especially as they relate to the 'long hundred'. He cited texts from the fifth, eighth, and even the seventeenth century.

Expanding on his previous talk, this lecture acquainted us with some of the methods researchers use to clarify the meaning of numbers in texts.

Using overhead transparencies to illustrate many points, Jens spoke about the confusion caused by words like 'hundred' or 'thousand' when translating early documents. It seems that our ancestors sometimes used such words to mean 100 and 1000, and at other times they meant 120 and 1200. Today these are called the *short* hundred or thousand, and the *long* hundred or thousand, respectively.

He explained that sometimes an author used expressions such as 'one hundred *twelve count*'. In these cases, it is easy for the translator to recognize that the number is one dozen decades, and not ten squared. Of course, the reason a writer would so modify the word 'hundred' is because the readers were aware of two different uses of the word and they would need the clarification to comprehend what was being said.

In other instances there are internal messages such as a reference in an Icelandic manuscript to "One hundred men served as soldiers; eighty stayed and forty left." (Jens pointed out that the translators not aware of the common use of the long hundred might mistakenly conclude that this was an arithmetic error!)

Similarly, in another place we find a citation that refers to fifteen score, one quarter of a thousand, or 300 — a confusing mixture of long thousand with short hundred. (A long thousand of 1200 is one dozen short hundreds or ten long hundreds).

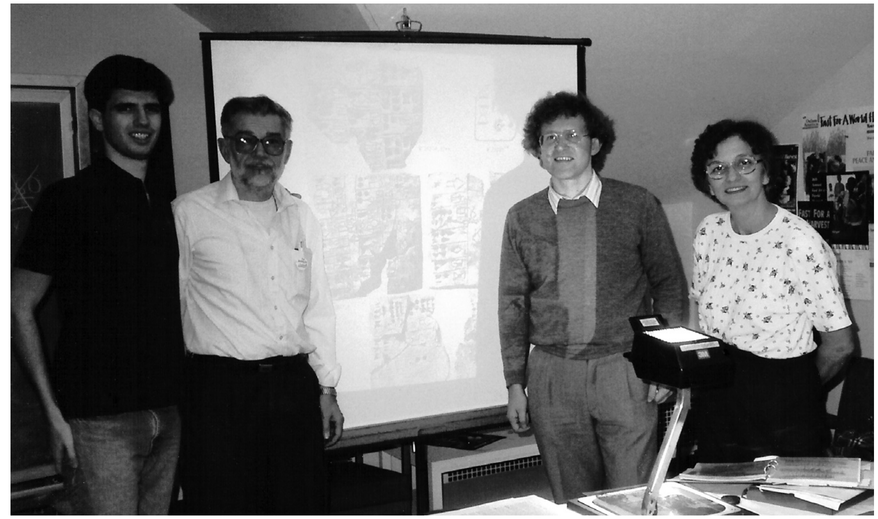
He explained some of the ways that historians attempt to decipher perplexing numbers in a document, including the fact that many times there may be no way to figure out which 'hundred' was meant.

One method that can be used to decipher the author's meaning is to find references to known quantities such as the 532-year Easter cycle in the Icelandic text, Alfredi:Rimtol. This cycle has been variously referred to as:

a.) *Four hundred twelve-counted, and two on the way to sixty*. This is equivalent to 4 × 120 + 50 + 2, as 'on the way to sixty' means counting in decades, passing the previous decade of fifty, but not yet reaching sixty.

b.) *Four hundred twelve-counted and forty and one dozen*, or 4 × 120 + 40 + 12.

c.) *Two on the way to forty on the way to six hundred*. 2 + 30 + 500. As before, 'on the way to forty' means 30, while 'on the way to 600' indicates 500.



David Rothstein, Prof. Gene Zirkel, Jens Ulff-Møller, and Prof. Alice Berridge shortly after the lecture on our forefathers' use of long hundreds and long thousands.

In most texts numerals were not used, but the words for numbers were written out. Notice that in (a) and (b) the long hundred is used, but in (c) it is the short hundred.

In another place a year is referred to as 'CCC nights and V nights'. In both England and in Scandinavia, the Roman numeral C was used ambiguously for both types of hundreds. In this case, it refers to 3 × 120 + 5 or 365 nights in a year. (Again, the translator who is ignorant of the use of C to sometimes represent the long hundred, might erroneously conclude that the author didn't know the correct length of the year.)

In the discussion that followed Jens' provocative talk, an analogy was made to our present use of the word 'ton'. There is the long ton, the short ton, the metric ton, and a couple of nautical tons that measure volume rather than mass!

Jens pointed out that logical people, particularly the Germanic tribes, would prefer to count and measure with a unit and then a unit squared, that is — either 10 and 100 or else 12 and 144. However, our ancestors were not always logical. Sometimes they were just practical. They couldn't calculate very well, and division was especially difficult. Ten as a unit most probably comes from the biological accident that we have ten fingers. However, one third of 100 or of 1000 is not convenient — hence they devised the long hundred and the long thousand.

Jens noted that our four fingers have three joints, and so hand counting is just as easy in duodecimals as in decimals with the added factor that one can count up to one dozen by using the thumb as a pointer. Of course, the reason that twelve is so often used is the fact that it has many factors, and hence division can be performed without using fractions.

All in all the lecture was informative and interesting, and we have learned to sympathize with some of the difficulties with which the historian is faced. We thank Jens for sharing the efforts of his doctoral research with us.

Alice Berridge & Gene Zirkel

"**Each one Teach One.**" *Ralph Beard, DSA Founder*

Menninger on the “Great Hundred”

Karl Menninger, a German historian, wrote *Zahlwort und Ziffer* in 1957–58, which was translated into English in 1969, and is now available from Dover as *Number Words and Number Symbols*. It is a magnificent, authoritative, and often-cited examination of the historical development and use of numbers across civilization. Among many other topics, Menninger examines the “great hundred” of 120 units, which he further explains was divided into dozens in northern Europe in the middle ages. Here are a few of his thoughts on the subject:

GREAT HUNDRED

[The great hundred] still measures quantities of fish, even in Germany, where it is used for other things as well. In Lübeck there was a *Hundert Bretter* = 120 items = 10 *Zwölfter* (“twelves”) or, as they said in Mecklenberg, 10 *Tult*. English differentiates between the *long hundred* of 120 units and the *short hundred* which has only 100. Today, however, we recognize “hundred” only in its meaning of 100.

The northern Germanic region, primarily Iceland, was the home of the great hundred; there *hundraþ* meant 120 in all monetary calculations and in designating military units, until the introduction of Christianity around the year 1000. Thereafter it came to stand for the small hundred of 100 in ecclesiastical and learned writings, and the two hundreds were sometimes, but not always, distinguished as the 10-(*ti-roed*) or the 12-(*tolf-roed*). A writer around the year 1250 once designated the 360 days of the year in the old manner: *III^c daga tolfroed* — “three hundred days by the 12-count,” and commented: “Nevertheless in the book language (Latin) all the hundreds are reckoned by the ti-count (*oll hundraþ tiroed*), which according to the proper count is *III^c tiroed ok LX daga*.”

We may well ask why the writer did not simply use the 100 for corresponding to *tio tiger*, “10 tens” — in this case “36 tens” — which our number sequence provides? Because it means ten tens in a 12-count gread hundred enumeration, but not 100 as such. We can understand this better if we read from an Old Norse tax roll:

“Whoever has property worth 1 ten shall pay 1 ell of *wadmal*;
“Whoever has property worth 2 tens … etc.; then
“Whoever has property worth *halft-hundraþ* (6 tens) shall pay 4 ells …
“Whoever has property worth *tio-tiger* (= 10 tens) shall pay 6 ells.”

Thus tio tiger meant 10 tens in the 12-count hundred system of counting, not an “independent” 100; *tio tiger ok þriu hunderaþ* is 460 = 10´10 + 3´120. […]

The standard of values was originally

1 *hundraþ silfrs* = 120 ounces of minted silver = 2400 ells of [*wadmal*] frieze-cloth, whence 1 ounce of minted silver = 20 ells of frieze-cloth…

THE NUMBER TWELVE AS THE BASIC UNIT OF THE GREAT HUNDRED

[…]Charlemagne’s monetary standard of the year 780, which had a lasting influence on medieval European coinage, clearly embodied the basic 12-unit:

(Latin) 1 *libra* or *talentum* = 20 *solidus* = 12 *denarius*;
1 pound of 20 shillings each of 12 pennies = 240 pennies, or
1 pound = 8 long (Bavarian) shillings of 30 pennies each = 240 pennies, but also:
1 pound = 12 ounces of 20 pennies each = 240 pennies.

In France the table of equivalents was:

1 *livre* of 20 *sou* (<*solidus*) of 12 *denier* (<*denarius*) each = 240 *denier*, whence the *sou* also came to be called a *douzain*, a “twelver”.

problem from last issue: *by Gene Zirkel*

problem corner

In a cryptogram, each letter has been replaced by a different letter. To solve the puzzle, one must recover the original lettering.

FTQ NQEF MDSGYQZF RAD NMEQ FIQXHQ AHQD NMEQ FQZ UE M XAAW MF FTQ RDMOFUAZMX QJBDQEEUAZ RAD 1/3 UZ NAFT NMEQE.

SOLUTION ON PAGE 29;

The various fines and penalties imposed under old Germanic law refer to a basic number: in the Alemannian, Bavarian, Friesian, Saxon, and Burgundian tribes this was 12, in Frankish law it was usually 10, and among the Lombards it was 12 for inflicting wounds or injuries and 10 for other infractions. In the Lex ripuaria, the code of the Ripuarian Franks, a stallion, a coat of mail and a hunting falcon were valued at 12 solidi; a helmet at half this amount, or 6 solidi; a sword with its scabbard at one third, or 4 solidi; a cow, a mare, and a sword without its scabbard at one fourth, 3 solidi; an ox, and a shield and lance at one sixth, or 2 solidi.

In *Lex salica*, the law code of the Salian Franks, a fine was once specified thus: *unum tualepti / sunt denari CXX / culpabilis iudicetur*, “the guilty one is sentenced to pay a Twelve, that is 120 pennies” — a document which again bears witness to the number 12 as the basic unit of the great hundred.

The importance of this number [twelve] in the daily lives of common people, in commercial transactions, and in legal affairs, is probably due to its easy divisibility in so many ways. The commonly used fractions of the *tylft* or the shilling could all be expressed in terms of whole numbers of pennies:

1	½	⅓	¼	¾	⅔	of a shilling
12	6	4	3	9	8	pennies

For this reason the north European *tylft* is an original, mative measure, and not one that was first brought in by way of the Carolingian coinage system. For, quite apart from its ready divisibility, it was also consistent with the Roman pattern in the table of ounces.

Karl Menninger’s work in this book explores the earliest human thoughts on number, from tally sticks to concepts of number groupings, from pre-Columbian America to the Fertile Crescent, to the far east. It is well-illustrated, and easy to understand. The only negative aspect of the 480-page 6½" × 9¼" paperback is that it largely ignores African experiences with number. Number Words and Number Symbols is highly recommended for anyone interested in how humanity uses numbers as tools.

MENNINGER, Karl. *Number Words and Number Symbols: A Cultural History of Numbers*. (P. Broneer, Translator). Mineola, NY: Dover, 1992. [1st ed. 1969, Cambridge, MA: MIT Press], pages 154–158, “Babylonian influence: Great hundred, The number twelve as the basic unit of the great hundred.”

VALIDATING THE
Dozenal Measure of Angle
by Michael De Vlieger

A key consideration in the establishment of any number base is measurement. What will be the basis of a new system of measurement? This is a large topic, because of this, let's focus on a relatively simple part of it. In this article, decimal units are preceeded by decimal figures, dozenal units by dozenal figures.

In general, there seem to be two modes of measuring the world. The first method sets up a base unit (the king's foot, the cubit, the number of wavelengths of light emitted by an unstable isotope) then concatenates this unit across the object to be measured until the number of units can be counted. The base unit is usually more or less arbitrary. In the first case, the civilizational number base will simply be applied, bundling the quantity according to that base. If the boat is the square of a dozen cubits long, and you're in a decimal culture, you've got 144 cubits; if a sexagesimal culture, you have 2 sixties and two dozen units. The challenge rests mostly in selecting a stable and reproducible base unit.

A second method takes into account that there is a more or less undeniable master unit (the day, the circle), then divides the master unit into aliquot parts which will serve as secondary units. Because we are dividing a master unit into aliquot parts, this method tends to use a highly factorable number to arrive at an arrangement friendly to the commonest fractions. In a later article we'll explore this more deeply.

For now, let's examine the easiest case of the second method. How should we divide the circle, given a dozenal number base? The Founders of the DSA explored this question, as have many others. In the *Manual of the Dozen System* on page 28; and 29;, we are presented with "The Duodecimal Circle", which then serves as the basis for that document's exploration of trigonometry. The unit circle is divided into a dozen "duors" (two hours of astronomical right ascension, 30°, or π/6 radians). The *duor* is divided into a dozen "temins" (ten minutes of astronomical right ascension, 2½°, or $^{\pi}/_{72}$ radians).

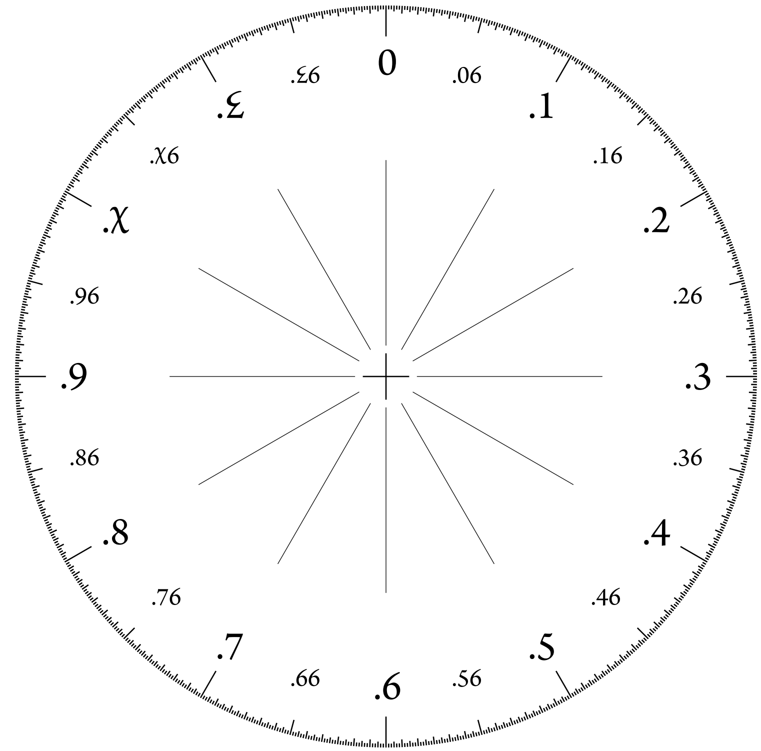
Being new to dozenals, I encountered the Duodecimal Circle on my own in 1983 and later in 2003, and felt skeptical about a purely dozenal division of a circle. In our decimal civilization, we use a three hundred sixty degree circle. Trigonometry classes and certain calculations use radians. Why use a dozenal measurement, when no one in history aside from us dodekaphiles seems to have arrived at a circle divided dozenally? On the internet a few years back, I read an assertion by a hexadecimal proponent that claimed dozenal proponents often construct studies that artificially uplift the dozen over ten or sixteen to further their case for dozenal. In the case of the division of a circle, we'll start from the beginning to try to avoid this bias. We'll see the most useful angles are geometrically determined. Their geometric utility demands human acknowledgement.

Dozenalists and proponents of hexadecimal and octal will agree that multiplication and division by factors of two are of utmost importance. We depart from accord when proponents of binary-power bases assert that two is the only important factor.

We begin by examining the simple fact that a circle can be produced by fully rotating a line segment in a plane about one of its points (see Figure 2A). Such a unit circle is the basis for all measurements of angle.

The first regular (two-dimensional) polygon many people will think of is the square, which can be made by propagating a line of a given length ℓ in a direction perpendicular to its length for the same length ℓ (See Figure 2C). The resultant figure has four equal sides with four equal, right angles. This figure can be rotated so that exactly 4 such figures contain the circle in Figure 2A—such a circle possesses a radius of ℓ. The square can be copied and tiled to fill up an infinite (Euclidean) two-dimensional plane.

The Duodecimal Circle



The Duodecimal Circle and Time

In the measurement of time and angle, the greatest simplicity is attained by using the circle and the day as the fundamental units, and the lesser division as duodecimals of these. In this way no conversion is necessary between minutes of time and minutes of angle. Time and longitude are expressed by the same number. The superscript (^{c}) can replace the degree (°) symbol.

UNITS

	(dozenal)			(decimal)	
;1^{c}	*is called the*	**duor**	=	2 hours	*or* 30°
;01^{c}		**temin**		10 minutes	2° 30'
;001^{c}		**minette**		50 seconds	12' 30"
;0001^{c}		**grovic**		4.16⅔ seconds	1' 2½"

KEY TRIGONOMETRIC FUNCTIONS

angle (θ)	0;1^{c}	0;16^{c}	0;2^{c}
sin θ	½	$^{1}/_{\sqrt{2}}$	$^{\sqrt{3}}/_{2}$
cos θ	$^{\sqrt{3}}/_{2}$	$^{1}/_{\sqrt{2}}$	½
tan θ	$^{1}/_{\sqrt{3}}$	1	$\sqrt{3}$

Figure 1, Remastered from material in the Manual of the Dozen System, *pages 28;–29;*

Graph paper demonstrates that the tiling of squares can be very handy. The Cartesian coordinate system with its familiar x and y axes, the street grids of cities like Chicago and Phoenix, the arrangement of columns in a big box store, all employ orthogonal arrangements of elements. Most of the built environment is based on the right angle. Our homes, offices, factories, streets, and cities commonly employ orthogonal geometry. Thus the right angle, a division of the circle into quarters, an angle of 90° ($\pi/2$ radians), is perhaps the most important division of the circle in everyday life.

The simplest regular polygon is a triangle, shown by Figure 2B. We can construct a triangle having three sides of equal length and three corners with the same angles, thus an equilateral triangle, simply by placing a compass at one end of a line, drawing a circle as in Figure 2A, then doing the same at the other end. We can draw straight lines from the intersections to both ends of the first lines to obtain a triangle. An equilateral triangle, if we were to copy it and cut it out, can be used to fill the circle in Figure 2A: exactly six equilateral triangles with a common vertex can fill the circle. In fact, we can tile the equilateral triangle to fill up an infinite plane just like the square. We can make equilateral triangle "graph paper". In trigonometry, the cosine of 60° ($\pi/3$ radians) is exactly ½. Because the equilateral triangle is the simplest regular polygon, because it can fill two-dimensional planes, and because precisely 6 equilateral triangles can fill a circle, it follows that such a 3-sided figure is important. Its geometry is thus important. The angles we've generated are all 60° ($\pi/3$ radians), $1/6$ of a full circle. So dividing a circle into six equal angles is an important tool.

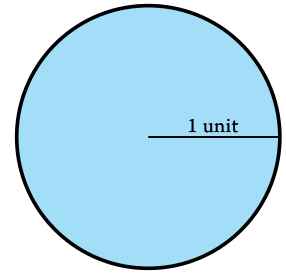
We can observe the importance of triangles in general and equilateral triangles in particular in our everyday society. Structural engineers design trusses, bar joists, and space frames with an equilateral arrangement, because the equilateral triangle is the stablest two-dimensional figure. Because its sides are equal, it can be mass-manufactured. The equilateral triangle is perhaps not as apparent as arrangements made with right angles (orthogonality), but it is important in the building of our everyday structures.

For the sake of this article, we'll call the equilateral triangle and the square "cardinal shapes".

Figures 2D and 2E show these cardinal shapes bisected (cut in half). There are three ways to bisect an equilateral triangle using one of its points, which are congruent if we rotate the triangle 120° ($2\pi/3$ radians). There are two ways to bisect a square using one of its points, which are congruent if we rotate the square 90° ($\pi/2$ radians). When we bisect an equilateral triangle, we obtain a right triangle with angles that measure 30°, 60°, and 90° ($\pi/6$, $\pi/3$, $\pi/2$ radians); see Figure 2F. When we bisect a square, we obtain a right triangle with angles measuring 45° and 90° ($\pi/4$ and $\pi/2$ radians); see Figure 2G. These bisections are important because they relate a corner of an equilateral triangle with the midpoint of its opposite side, or the diametrically-opposed corners of a square to one another. In trigonometry, the sine of 30° ($\pi/6$ radians) is exactly ½. Thus, 30° and 45°, one dozenth and one eighth of a circle, respectively, are of secondary importance. We'll call the bisected equilateral triangle and the diagonally bisected square the "bisected cardinal shapes" for the sake of this article.

Figure 2H shows that the difference between the bisecting angles of the cardinal shapes is 15° ($\pi/12$ radians), one two-dozenth of a circle. Using 15° or one two-dozenth of a circle as a snap-point, one can construct any incidence of the bisecting angles of a square or equilateral triangle. In fact, before the advent of computer-aided design and drafting (CADD), draftsmen commonly used a pair of "45°" and "30°–60°" drafting triangles (see Figure 2J), along with a T-square or parallel bar precisely to obtain the common angles which are two dozenths of a circle. Thus, it is not by dozenalist design but sheer utility that the two-dozenth of a circle, or 15° ($\pi/12$ radian) angle is deemed important.

Figure 2A. The unit circle, a circle with a radius of one unit of measure. This is the basis for this study of important angles.



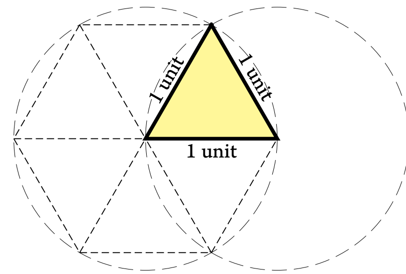


Figure 2B. The equilateral triangle.

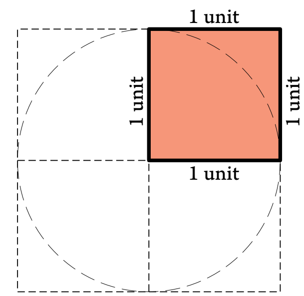


Figure 2C. The square.

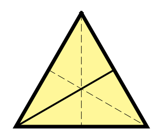


Figure 2D. Bisecting the equilateral triangle.

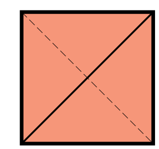


Figure 2E. Bisecting the square.

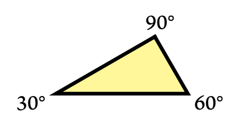


Figure 2F. One half of the equilateral triangle.

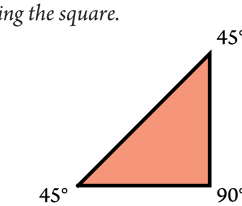
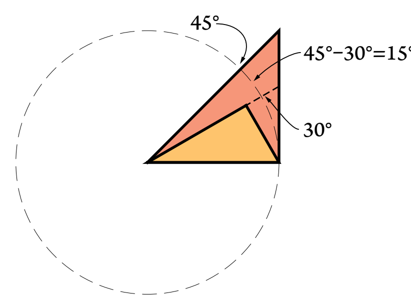


Figure 2G. One half of the square.



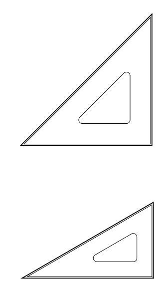


Figure 2H. Difference between Figures 2F and 2G.

Figure 2J. Drafting triangles.

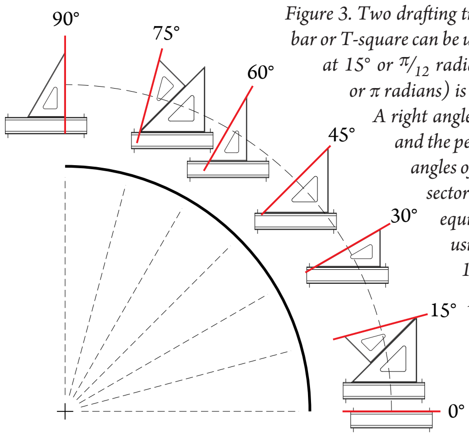


Figure 3. Two drafting triangles (the bisected cardinal shapes) and a parallel bar or T-square can be used to construct all basic angles (the two dozen angles at 15° or $^{\pi}/_{12}$ radian increments). The straight angle at 0° or 180° (0 or π radians) is constructed using the parallel bar or T-square alone. A right angle can be produced using the parallel bar or T-square and the perpendicular side of either of the drafting triangles. The angles of an equilateral triangle (60° or $^{\pi}/_{3}$ radians) or the bisector angles of either the square (45° or $^{\pi}/_{4}$ radians) or the equilateral triangle (30° or $^{\pi}/_{6}$ radians) can be produced using a single draftsman's triangle on a parallel bar. The 15° ($^{\pi}/_{12}$ radian) or 75° ($^{5\pi}/_{12}$ radian) angles can be produced with both triangles on the parallel bar. If the drafting triangles are flipped or mounted on the bottom of the parallel bar or T-square, the angles in the other quadrants of the circle can be drafted.

To be sure, other regular polygons can be drawn. The pentagon appears in regular three dimensional polyhedra, in the dodecahedron and the icosahedron, and in their symmetries. Geometrically, it is an important figure, however it cannot tile two dimensional space, and is not in common use in everyday life. The fact that most Americans can name precisely one building that is shaped like a pentagon contributes to the case that pentagonal arrangements are a curiosity because they are rare. The hexagon is important: we can see in Figure 2C that the outline of the group of triangles inscribed in the left circle is a regular hexagon. Thus the hexagon can tile two dimensional planes. Role playing games in the eighties employed the hexagonal grid system. There are some curious places (the Nassau Community College campus, the Price Tower in Bartlesville, Oklahoma, and other Frank Lloyd Wright buildings) which are arranged in a triangular-hexagonal geometry. The geometry of the hexagon is corollary to that of the equilateral triangle. There certainly are many regular polygons in use in human civilization and apparent in nature, but the commonest and most important geometries appear to be linked to the equilateral triangle and the square.

Other angles are important. On a map, we commonly hear "north northwest" or "east southeast", the sixteen partitions of the circle. The 22½° ($^{\pi}/_{8}$ radian) angle, one sixteenth of a circle, is important, perhaps more significant than 15° divisions in cartography. The sixteenth of a circle shouldn't be ignored for this reason.

The decimal division of the circle into thirty dozen degrees is a quite handy tool (See Figure 4A and 4B). Each of the two dozen angles related to the equilateral triangle and the square (let's call these two dozen angles "basic angles") are resolved in the system of degrees without fractions. The system neatly accommodates fifths and tenths of a circle, although these are comparatively rarely used.

Under a dozenal system, we may discover that using a strictly dozenal division of the circle, perhaps using "temins" (perhaps abbreviated ^{t}) for convenience, neatly accommodates all the basic angles as well as the sixteenths of a circle without fractions (See Figure 5A and 5B). The basic angles are simply multiples of 6^{t}. The sixteenths of a circle are multiples of 9^{t}. If we desire to "unify" the basic and the sixteenth-circles into one system, we might regard 3^{t} (7½°, $^{\pi}/_{24}$ radian) as the dozenal "basic angle". We surrender the fifths (pentagonal symmetry) and tenths to repeating fractions, but maintain a fairly simple measure of the commonest angles. A strictly dozenal division of a circle, as presented by our Founders, thus appears sound and sufficient for everyday use.

A further exploration of division-based measurement will appear in a coming issue.

Notes for Figures 4 & 5: The unit circles shown here are drawn so that the black tickmarks are the length of the reciprocal of their denominator. Thus the tickmark at ½ circle is ½ unit long, those at ⅓ and ⅔ are ⅓ unit long, and those at ¼ & ¾ are ¼ unit long. Red lines are used to indicate fractions of a circle which are multiples of the reciprocals of powers of two such as quarters and eighths, with the heaviest line indicating ½. Gold lines indicate multiples of the reciprocals of multiples of three, with the thirds receiving the heaviest lines. Blue lines indicate multiples of the reciprocals of the multiples of five, with fifths receiving the heaviest line. In the enlarged quadrants at right, the dashed line reminds us that the 60° angle is one which generates an equilateral triangle.

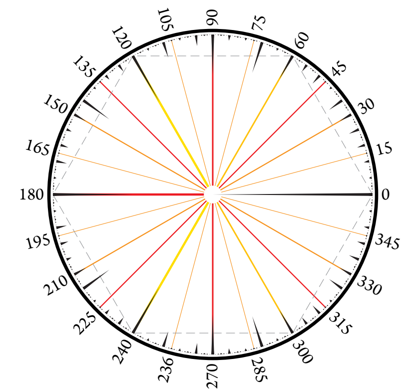


Figure 4A. The degree system used in decimal civilization. The system was set up by our forefathers under a sexagesimal number base. The system survives to this day and continues to be used perhaps because all the basic angles are represented by decimal semiround or round numbers.

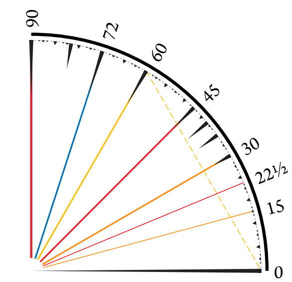


Figure 4B. A closer examination of some key angles under the degree system. The right angle and the square bisector angle are 90° and 45° respectively. The equilateral angle and its bisector are 60° and 30° respectively. The difference betweeen the square and equilateral bisectors is 15°. The sixteenth of a circle is 22½° or 22°30′. The fifth of a circle measures 72°

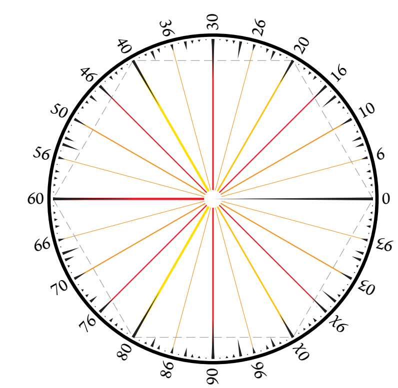


Figure 5A. The circle divided into temins, a gross temins to a full circle. The unitary relationship to the full circle is maintained, the notation for each of the basic angles is simpler, and the number of radians can be quickly determined by dividing the temins by six dozen, then multiplying by π.

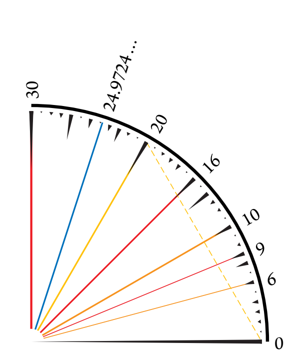


Figure 5B. A closer examination of some key angles expressed as temins. The right angle and the square bisector angle are 30^{t} and 16^{t} respectively. The equilateral angle and its bisector are 20^{t} and 10^{t} respectively. The difference betweeen the square and equilateral bisectors is 6^{t}. The dozen-fourth of a circle is simply 9^{t}. The fifth of a circle is 24;9724… ^{t}, a repeating digital fraction.

Twelfty, or Adventures with Alternating Bases

The Duodecimal Bulletin interviews Wendy Y. Krieger

Preface

Wendy Krieger is a retired Australian with a degree in physics, a penchant for geometry of higher spaces, and the number she calls “twelfty”. One of my own periodic interests involves polychora, the figures of four-dimensional space—trawling for interesting tidbits on the internet eventually brought me to Ms. Krieger’s “polygloss” segment of her website[1]. What stood out from the nomenclature associated with these higher figures is that in places[2], she referred to “twelfty”, which turns out to be base ten-dozen[3]! This prompted me to contact her in the middle of last year to see if she might send some of her thoughts about such a large base, and why she considers the long hundred to be of use to her. Ms. Krieger sent a few notes and has corresponded on the subject. What follows is an account of Ms. Krieger’s rationale, which I have backed up with some research and illustration. We dozenalists may not fully agree with Ms. Krieger’s observations, but we are accustomed to “Excursions in Numbers”, and this is a grand excursion.

> ***“Pure duodecimalization appears to ignore that multiplication and division are different processes in the human mind”***

A few notes about Ms. Krieger’s use of “twelfty”. Wendy “shopped” for a number base which would facilitate the recognition and manipulation of assorted vulgar fractions whose denominators were multiples of small prime numbers. An example of a reciprocal of a simple prime would be ½, ⅓, 1/5, or 1/7. She methodically examined the digital periods of reciprocals of small primes in many bases, looking beyond the prime divisors to the coprime reciprocals, for the shortest repeating digits across the simplest prime reciprocals[4]. She was looking for simple repeating fractions. Table 1 illustrates “preferred intervals” of the reciprocals of the eight most simple and commonest primes across many bases. Dozenal features 2 regular numbers[5] deriving from primes (½, ⅓), which dozenalists recognize as 0;6 and 0;4. The dozenal expansions for 1/Ɛ and 1/11; aren’t bad at $0;\bar{1}\ldots$ and $0;\overline{0Ɛ}\ldots$ with a period length of 1 and 2 respectively. Dozenal doesn’t handle 1/5 or 1/7 very well at all, requiring the maximum period length for both. Base ten dozen features 3 prime regular numbers[5] (½, ⅓, 1/5), but also short periods for the reciprocals of 7, Ɛ, 11;, and 15;!

Having recognized base ten-dozen as useful, Ms. Krieger looked to the ancients and how they regarded quantity. Our modern society is based on a uniformily-applied decimal system. We count individual packages in decades and hundreds, in multiples of positive integral exponents of ten. Our mechanized society prefers percentages, dividing by multiples of the negative integral exponents of ten. Base ten is applied uniformly, across the decimal point today, but the Romans and the Greeks grouped multiples by ten, and divisions by highly factorable numbers. There is evidence both cultures counted in a “biquinary” fashion; once they counted to five, they noted it, continuing to ten thereafter, then carrying a one to the next place. Evidence of this can be seen in Roman numerals. The Babylonians used base five dozen for both multiples and divisions of a unit. They split their compound digits into a decade-figure and a unit-figure. Ms. Krieger was attracted to the ancient compound digit, which she recognized as an “alternating base”.

For the sake of this article, wherein we consider alternative bases, we’ll call the decimal point the “unit point” and the uniform application of a base a “uniform base”. We’ll refer to “decimal expansion” for bases other than decimal as “digital expansion”, “decimal fractions” as “digital fractions”.

Let’s take a closer look at Ms. Krieger’s case for alternating base ten-dozen. *the Editor*

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The uniform expression of any integer base *r* necessitates a set of *r* unique digits all less than *r*. These single-character, integer digits range between {0, (*b* − 1)} inclusive. The basic arithmetic tables are simply arrays of r^2 elements. We may expand fractions $1/n$ in base *r* “digitally”, registering multiples of negative powers of *r*. The fraction 1/5 expressed decimally is written 0.2, while the dozenal expression is the recurring digital fraction $0;\overline{2497}\ldots$

Any number *r* might serve as a base. In practice, the human limitations of memorizing the full arithmetic arrays tends to limit consideration of bases around the magnitude of decimal, such as octal, dozenal, or hexadecimal. Waiving the mnemonic restrictions on the scale of multiplication tables allows one to consider large bases like five- or ten-dozen, or even decimal 1680. Free of the constraints of human memory, one can use a modern calculator in place of arithmetic tables. We might use base 5832. (18^3) or 20736. (12^4) or some other power of a lesser number to do precise calculations in base 16; or 10;, computing three or four of the digits of these smaller bases at a time.

Over the years, Ms. Krieger has used many bases, even to the extent of devising measurement systems for them. This is more to understand the nature of number, since measurement and number are often entwined. One interesting scale she notes is the Rankine temperature scale in dozenal. In such scale, water freezes at 350;^{R} and boils at 480;^{R}. The normal thermometer runs from 300;^{R} (i.e. −28° F = −33 C) to 400;^{R} (116° F = 46.7 C), making cold as a low number and hot a high number in the same gross. The Rankine scale is the basis of her own “gorem” temperature scale in “twelfty” (see Figure 8).

We can regard ordinary numbers has having an integer part and fractional part. If we regard the

	2	3	5	7	Ɛ	11	15	17
	1	2	4	6	χ	10	14	16
6	•	•	1	1	M	M	M	9
8	•	M	M	1	M	4	8	6
χ	•	1	•	M	2	6	M	M
10	•	•	M	M	1	2	M	6
12	•	M	2	•	M	1	M	M
13	M	•	•	1	5	M	8	M
14	•	1	1	3	M	3	2	9
16	•	•	M	3	M	4	1	2
18	•	M	•	2	5	M	M	1
19	M	•	1	•	2	4	4	M
20	•	•	2	M	M	M	M	9
24	•	1	M	•	M	M	M	9
26	•	•	•	3	M	6	4	3
28	•	M	M	3	2	M	8	M
30	•	•	1	1	5	6	8	9
36	•	•	M	•	5	3	8	9
40	•	•	M	2	5	3	M	M
50	•	•	•	3	5	4	8	M
60	•	•	M	3	M	M	4	M
80	•	•	1	M	M	4	M	1
83	M	•	2	1	•	4	M	9
χ0	•	•	•	1	2	3	1	9
156	•	•	•	•	1	M	M	1
260	•	•	•	M	M	3	M	2

Table 1. “Preferred intervals” of repeating digital representations of the reciprocals of small prime numbers in various bases. The small primes are listed at the top, increasing rightward from the smallest (2) at the leftmost column. Bases considered appear at the head of the rows. A bullet (•) signifies a regular digital fraction in that base, which has a simple terminating fraction. Example: decimal 1/5 = .2. The length of the set of repeating digits is given for each prime nondivisor in each base. Examples: decimal ⅓ = $.\bar{3}$, period of 1, or dozenal 1/5 = $;\overline{2497}$, period of 4. An “M” signifies that the digital fraction of a given prime reciprocal is a cyclic number, with the maximum period length. Thus the dozenal example of 1/5 is noted in the table as “M”. The maximum period lengths (the lengths of cyclic numbers for the prime reciprocals) appears below the primes at top.

n	$\bar{n}$	n	$\bar{n}$	n	$\bar{n}$
2	30	16	3,45	45	1,20
3	20	18	3,20	48	1,15
4	15	20	3	50	1,12
5	12	24	2,30	54	1,6,40
6	10	25	2,24	1	1
8	7,30	27	2,13,20	1,4	56,15
9	6,40	30	2	1,12	50
10	6	32	1,52,30	1,15	48
12	5	36	1,40	1,20	45
15	4	40	1,30	1,21	44,26,40

Table 2. Standard table of reciprocals used in ancient Mesopotamia. The notation is sexagesimal, represented by decimal representation[X].

significant digits of a number expressed in scientific notation, we can extract a *significand*[6], an integer formed by the significant digits, expressed without leading or trailing zeroes, the exponent, nor the base. The significand of decimal 12.5 is 125, and of 600, 6.

The divisors d of of base r can be paired in such a way as to yield r when multiplied together: $r = d \times d_1$[7]. The significand of r in base r is 1. Melville (2005.) writes that ancient Mesopotamian civilizations expanded this principle to any pair he termed "reciprocals"[8] (which we'll call "reciprocal divisor pairs"), consisting of positive integers $n\bar{n}$ whose product is a power of sixty[9]. Figure 1 illustrates reciprocal divisor pairs $n\bar{n}$ in a typical standard table used by the ancient Mesopotamians for easy reference[X], here presented in decimalized sexagesimal digits. A decimal example of a reciprocal divisor pair is 8. × 125. = 1000., the significand of 1000. being 1.

The products of a divisor d and a digit t coprime to the base r yields products dt whose inverses $1/dt$ feature the same digital expansion as $1/t$. A dozenal example of this can be seen in the multiples of 7 as denominators of fractions. These multiples of 7 have fractional parts that feature some number of zeros preceding the repeated string of digits generated by ¹/₇:

¹/₇ = 0;186 X35 186 X35…
¹/₁₂; = 0;0X3 518 6X3 518…
¹/₁₉; = 0;06X 351 86X 351…
¹/₂₄; = 0;051 86X 351 86X…
¹/₂Ɛ; = 0;041 455 9Ɛ3 931…

The fact that the repeated string of digits characteristic to ¹/₇ is observed in the (fractional) reciprocals: that is, the denominator evenly divides $7 \times 10;^n$ for some exponent n.

Our Forefathers Used Mixed and Alternating Bases

The counting board (also known as an "early abacus"), the progenitor of the (modern) abacus, a table on which one moves counter stones about to calculate[Ɛ]. Columns are drawn upon the surface, upon which one manipulates the counters. When a counters exist on a column, one replaces these a counters with a single counter in the next column. The Salamis Tablet, a stone counting board found on the Greek island of the same name, had two zones for representing numbers (see Figure 1.) Stephenson (2010) argues that the Babylonians used the main, lower zone of the counting board for computing, representing a significand starting at the top line. The top line then functioned as the unit point of the number. He writes that the upper zone was used to calculate the exponent of the significand, so that the counting board represented a number in a format similar to scientific notation, except that the ancients positioned the unit point in front of the significand, rather than after the first digit of the significand as is the practice today. A decimal example: 2011. would be represented like .2011 E+4 rather than 2.011 E+3[11].

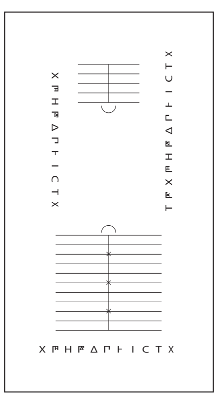


Figure 1. The Salamis Tablet, an example of an ancient counting board. There are two zones, an upper and a main (lower) zone, seen here as a group of horizontal lines. Letters which Menninger (1969) writes "can be identified as early Greek numerals and also as denominations of coins" appear on three sides of the counting board.[10]

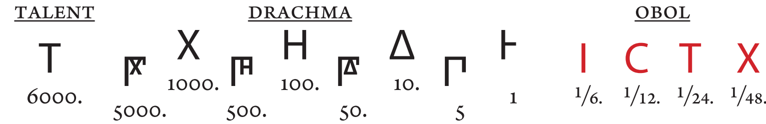


Figure 2. Greek symbols on the Table of Salamis. Symbols at left represent multiples, in decimal denominations. The leftmost unit is the talent, while the symbols printed higher under "drachma" represented pure decimal powers. The lower symbols represented 5 times the decimal power symbol immediately to the right. The four symbols at right represent, from left to right, the obol (1/6 drachma), the demi-obol (1/12 drachma), the quarter-obol (1/24 drachma), and the chalkos (1/8 obol, 1/48 drachma)[13].

Stephenson posits that the Greeks inherited the couting board technology from the Babylonians[12], which were uniformily sexagesimal. The later cultures simply converted the uniform sexagesimal arrangement of the Mesopotamian counting board to their own decimal-multiple, dozenal-division arrangement[13].

The ancient Greeks inscribed the index in Figure 2 on their counting boards, tallying whole units using denominations of 1×10^n and 5×10^n *drachma*. The Greeks divided the *drachma* into 6 *obol*. Three smaller denominations were ½, ¼, and ⅛ *obol*[14]. The Romans counted multiples in units of 1×10^n and 5×10^n, similar to the Greeks. These were represented by the familiar Roman Numerals. When the Romans divided their basic units of measure and money, they used the duodecimal *uncia*[15]. The Roman hand abacus shown in Figure 3 featured decimal multiples prominently on the left side of the device, but dozenal divisions to the right. So the ancient Greeks and Romans appear to be accustomed to using decimal when dealing with multiples, and highly composite divisions of a unit: our forefathers were accustomed to dealing with mixed radixes.

Multiples and Divisions

Ms. Krieger observes that human societies have tended to deal with multiples and fractions in separate ways. The application of counting board calculations, especially in the Roman and Greek traditions, contributes to this idea.

Wendy also notes the historical tendency to deal differently with multiples and frac-

Figure 3. A Roman pocket abacus, which enabled calculation with mixed radix numbers. The integer-part of the number in question would register in powers of ten to the left of the dashed line. Right of the line, the mantissa or fractional part would be computed in highly factorable denominations. Uncia, *twelfths of a unit, were counted in the first column to the right of the dashed line. The rightmost column computed dozenal parts of the* uncia[7].

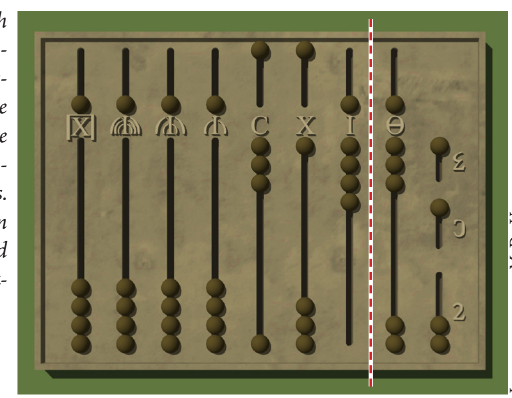


Illustration by M. De Vlieger

tions may be linked to the way we think. Butterworth (1999) writes that multiplication and division are handled in different parts of the brain [16]. A person affected with some brain damage may have lost the part that deals with multiples but can still handle division. Moreover, only one part of the brain appears to be involved in forming multiples, while several different parts are each able to handle division on its own.

Regarding the treatment of measure in the human mind, two separate techniques of reckoning multiples and divisions of a unit can be recognized. The "multiples technique" covers reckoning multiples of integral units, while the "divisions technique" governs fractional parts of a unit. Ms. Krieger notes that we may extrapolate these techniques to cover the "territory" of the other. "Sub-multiples" may be extended below the unit point to measure the fractional part by counting multiples of negative powers of the base. This may be seen in the decimalization of fractions. "Super-divisions" may be extended above the unit point to measure integral units, by dividing positive powers of the base.

Wendy muses on the following observations regarding dozenal. Dozenalists appear to have presumed that the uniform base is the optimum approach to representation of both multiples and divisions of units. Decimal extension below the unit point fails to yield convenient and efficient divisions of the unit; this is something that dozenalists recognize. The decimal is in play outside of its territory. Consider, perhaps the highly divisible dozen as "out of its element" as well. Dozens and grosses appear to be super-divisions, regarding the multiples of integral units as highly divisible as the *unciae* or twelfths. In this way, pure duodecimalization appears to ignore that multiplication and division are different processes in the human mind.

The Abacus and Alternating Bases

The Greek notation on the left side of Figure 2, ignoring the talent, deals in two denominations, a set of pure decimal powers (Χ, Η, Δ, Ͱ) and a second intercalated set of five times the decimal powers (𐅇, 𐅆, 𐅅, Π). The left side of the abacus shown by Figure 3 is arranged in eight columns, with a smaller slot positioned above a larger slot. The Roman abacus represented numbers in a given column in the lower slot up to four. When one had five, the lower slot was cleared and the upper slot was marked. Numbers n greater than five had the upper slot marked, and the requisite number ($n - 5$) indicated in the lower slot. At ten, a single unit in the lower slot to the left of the original column would be marked, and both slots in the original column would be cleared. The slots in the Roman abacus and the intercalated ranks of Greek numerals are instances of an alternating base. In counting, one climbs to 5, marks it, then continues to ten, carrying a one to the next decimal rank.

The ancient civilizations did not possess a zero [17]; however one does not need a zero if there is already some other way of indicating the order of magnitude of a digit. Wendy's example follows:

"5 hours 22 seconds"	is the same as	5 H. 0 M. 22 S.
"5 hours"	is the same as	5 H. 0 M. 0 S.
"22 seconds"	is the same as	0 H. 0 M. 22 S.

The distinction between "twenty eight" and the colloquial phrase "twenty-oh-eight" comes from this particular form. The "oh" part tells us that the eight is not at a magnitude directly smaller than the "twenty", (i.e. 28.). Instead, it pairs digits in the number and signifies there is a higher digit-pair with a vacant low rank and the least significant digit-pair with a vacant high rank (eg 20¦08). Thus we can express a vacant magnitude without a zero symbol and still express a number intelligibly.

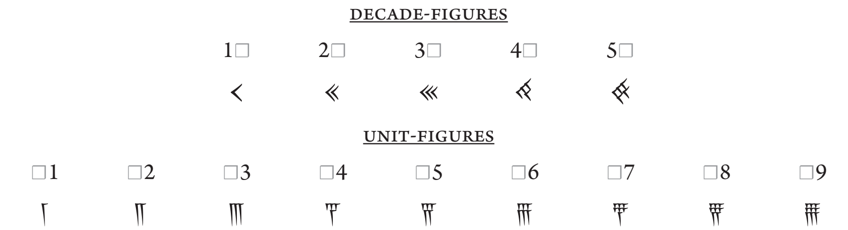


Figure 4. The 5 decade-figures and 9 unit-figures that compose the sixty digits of the ancient Mesopotamian sexagesimal system. The decade-figures may appear alone; in this case, the decade-figure simply stands for a round decade. 4□, by itself, signifies digit-forty, or 𒐏, and not digit-four or 𒐉. The unit-figures may likewise appear alone and communicate a multiple of units lesser than a decade. □5, alone, signifies digit-eight, or 𒐍, and not digit-fifty or 𒐐. A paired decade- and unit-figure composes intermediate digits. Thus, 4□ and □5 together compose digit-forty-five or 𒐏𒐊. Figures paired in the opposite order, thus □5 and 4□, must be interpreted as separate digits (digit-five, digit-forty), as must two decade-figures (say 4□, 1□) or two unit-figures (e.g. □5, □7). The digit-five digit-forty or 𒐊𒐏 represents the significand $^5/_{60} + ^{40}/_{60^2}$, the exponent given by context. ***

Thus, the ancient Greeks and Romans counted and manipulated multiples in bi-quinary decimal (decimal based on two cycles of five units each per digit), and manipulated fractions using highly divisible number bases. Ms. Krieger laments that today's uniform decimal base ignores the alternating base notion of the upper and lower slots on the left side of the abacus shown by Figure 3.

The Ancient Mesopotamian Sixty

The Babylonians inherited sexagesimal numeration from the Sumerians [18]. The Babylonian cuneiform system, shown by Figure 4, uses five decade-figures and nine unit-figures to compose a sexagesimal digit. We'll refer to the decade-figures conceptually as a numeral followed by a box, e.g. the decade-figure signifying "four decades" we'll show as 4□. The unit-figures will feature a box preceded by a numeral, e.g. the unit-figure meaning "five units" we'll refer to as □5. The decade-figures could stand alone and communicate the value of a pure decade: the decade-figure 4□ or 𒐏, by itself, communicates sexagesimal digit-forty. The unit-figure alone communicates sexagesimal digits with values lower than digit-ten: the unit-figure □5 or 𒐊 signifies sexagesimal digit-five. When a decade-figure and a unit figure appear in sequence, they are interpreted as a pair and compose an intermediate sexagesimal digit greater than digit-ten. Decade-figure 4□ or 𒐏 followed by unit figure □5 or 𒐊 compose the sexagesimal digit-forty-five or 𒐏𒐊. Confusion arises when a Babylonian scribe marks 4□ □5 or 𒐏 𒐊, meaning "forty sixties and five ones", and fails to leave a space wide enough to distinguish it from digit-forty-five [19].

Ms. Krieger posits that the precursor to the sexagesimal system regarded the decade-figures instead as fraction symbols, where 1□ or 𒌋 represents $^1/_6$, 2□ or 𒎙 represents $^1/_3$, etc. so digit-forty-five or 𒐏𒐊 could be either read digitally as [$^4/_6$][5] or [40][5], context providing the actual value of the numerals. Neugebauer and Menninger contribute that interpretation of the scale of the significand and the convertibility between common fractions and whole numbers of measure were important to the Babylonians [1X]. The alternating decade-figures and unit-figures arrangement in the digits of cuneiform sexagesimal notation Ms. Krieger interprets as an alternating base-6 and decimal notation.

Wendy asserts the sexagesimal system appears intended to avoid division. As evidence she supplies the following:

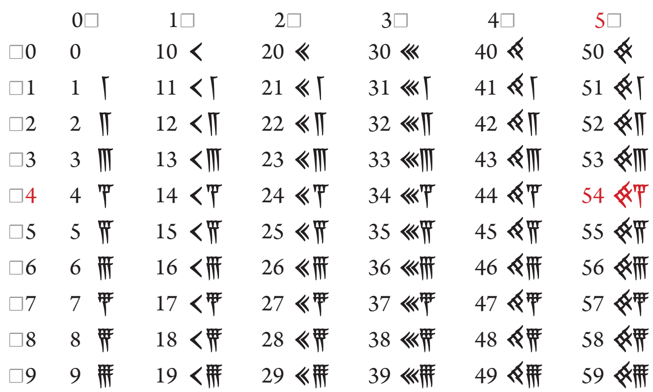


Figure 6. The ancient Mesopotamian sexagesimal digits were composed of five decade-figures and nine unit-figures. With these fourteen figures, sixty unique digits were composed. Krieger's "twelfty" notation uses a pair of figures to represent a digit. One may find the corresponding example of an ancient Mesopotamian digit by finding the decade-figure in the header row, then using the leftmost column, finding the unit-figure. By running a finger to the column of the high rank figure, the unique sexagesimal digit can be identified. Example: decade-figure 5□ and unit-figure □4 renders the digit-54: 𒐐𒐘.

- Many Babylonian arithmetic tables are reckoners of some integer *x* multiplied by the whole numbers 1 through 19, then each decade under sixty [1Ɛ]. This eliminates the need for a complete sexagesimal multiplication table. She remarks that similar decimal "ready reckoner" tables were in use until the advent of electronic calculators.
- The Babylonians developed opposition tables, or tables of reciprocals [X], arranged according to significands, an example shown by Table 2.
- There are reckoner tables for *x*, where *x* corresponds to divisors of higher powers of sixty, like decimal 1/81 (sexagesimal 44,26,40). [20]
- Neugebauer refers to the seven brothers problem, which amounts to finding 1/7 sixty-wise, concluding 08,34,16,59 < 1/7 < 08,34,18. [21]
- Zeros are significant in initial and medial positions, but not trailing positions. A number like 0,0,1 is not "1" (e.g. 1 hour), but 1/3600 h = 1 second. [22]

Ms. Krieger observes that the sexagesimal system continues to be used today in the sciences of astronomy and geometry. Our watches count five dozen minutes to the hour, five dozen seconds to the minute. The circle is divided into 360 degrees, which equals 6 iterations of the sixty degree equilateral triangle; these degrees themselves are divided into sixty arc-minutes and sixty arc-seconds [23]. We continue to divide the heavens into arcseconds, although these arcseconds are now divided decimally. Similarly we now count hundredths of a second in a marathon or an Olympic swim, and science uses milli-, micro- and nanoseconds. Today's culture is still fascinated with the Babylonian system, evidenced by a 23 November 2010 *New York Times* article [E].

Why "Twelfty"?

As stated in the preface, Wendy examined number bases for what Ore (1948)refers to as "determining denominators that yield short periods" of recurring digits in their digital fractional expressions. These short periods she called "preferred intervals". She applied the formula $(r^n - 1)$ [24], obtaining results similar to Table 3, and getting a general picture across many bases *r* for the smallest primes as shown by Table 1. Twelfty, like decimal, base

DOZENAL n	2	3	5	7	Ɛ	11	15	17	Others
1	✓	✓			●				
2	✓	✓			•	●			
3	✓	✓			•				111
4	✓	✓	●		•	•			25
5	✓	✓			•				11111
6	✓	✓		●	•	•		●	111

"TWELFTY" n	2	3	5	7	Ɛ	11	15	17	Others
1	✓	✓	✓	●			●		
2	✓	✓	✓	•	●		•		
3	✓	✓	✓	•		●	•		791
4	✓	✓	✓	•	•		•		8401
5	✓	✓	✓	•			•		5X0404X1
6	✓	✓	✓	•	•	•	•		8321, 791

Table 3. Prime factors present in $(r^n - 1)$ *for dozenal and "twelfty", where the value of n is incremented from 1 to 6. The checkmarks remind us that the prime factors above are divisors of r, and thus have regular fractions. A bullet (•) marks the smallest value of n that is divisible by the prime. This value is also the length of the recurring fractional period associated with that prime in base r. Other instances where the same prime appears in the factorization of* $(r^n - 1)$ *for higher values of n are grayed out.* [24]

twenty-one, and base ninety-nine, has a marked "preferred interval" system. This means that there is a neighboring integer furnishes small prime divisors not represented in the prime factorization of the base. For example, the prime factorization of ten is {2 · 5}, while the prime factorization of one million minus one is $\{3^3 \cdot 7 \cdot Ɛ \cdot 11; \cdot 31;\}$. In the case of twelfty, its prime factorization is $\{2^3 \cdot 3 \cdot 5\}$, while twelfty squared minus one is $\{7 \cdot Ɛ^2 \cdot 15;\}$. This means these bases will resolve a larger number of possible denominators with shorter recurring periods, facilitating factorization of these denominators.

	DECIMAL	TWELFTY	DOZENAL
1/7	0.142857142857	0:17 17 17 17	0;186X35186X35
1/11	0.090909090909	0:10 X9 10 X9	0;111111111111
18/77	0.233766233766	0:28 06 28 06	0;297Ɛ46297Ɛ46

Dozenal supplies "preferred intervals" of length 4 for 1/5, 6 for 1/7, 1 for 1/Ɛ, and 2 for 1/11;. The fifth, and all fractions with denominators having prime factors 2, 3, and 5, are regular in base ten-dozen. supplies preferred intervals no longer than three digits for 1/7 through 1/15. "Twelfty" handles fractions much better than dozenal!

	TWELFTY	PURE 120.	DECIMAL	DOZENAL	ALT. 60.	PURE 60.
1/7	:17 17 17	[illegible]	.142857	.186X35	:08 34 17	[illegible]
2/7	:34 34 34	[illegible]	.285714	.35186X	:17 08 34	[illegible]
3/7	:51 51 51	[illegible]	.428571	.5186X3	:25 42 51	[illegible]
4/7	:68 68 68	[illegible]	.571428	.6X3518	:34 17 08	[illegible]
5/7	:85 85 85	[illegible]	.714285	.86X351	:42 51 25	[illegible]
6/7	:X2 X2 X2	[illegible]	.857142	.X35186	:51 25 42	[illegible]
	1(2/2)	1(1)	1(6)	1(6)	2(3)	2(3)

Table 4. Examination of the multiples of one seventh, the smallest prime which is coprime to bases 10, 12, 60, and 120. Only the mantissas are considered for Krieger's twelfty notation, a pure base 120, decimal, dozenal, an alternating base 60, and a pure sexagesimal notation. Since seven is prime, and coprime to all the bases examined, the reciprocal of seven is a purely periodic irregular digital fraction. The twelfty notation groups the high-low rank pairs, separating the groups by a space. Each pair of figures represents one base-120 digit. Thus the left (high) rank in the pair has a dozenal and the right (low) rank has a decimal representation. Bars placed above a digit or rank represent part of the repeated fraction. The last row lists the number of families of mantissas, with the length of the recurrent period in parentheses.

	DOZENAL	TWELFTY	DECIMAL
2	**;6**	**:60**	**.5**
3	**;4**	**:40**	.$\overline{3}$…
4	**;3**	**:30**	**.25**
5	;$\overline{2497}$…	**:24**	**.2**
6	**;2**	**:20**	.1$\overline{6}$…
7	;$\overline{186\chi35}$…	:$\overline{17}$…	.$\overline{142857}$…
8	**;16**	**:15**	**.125**
9	**;14**	**:13 40**	.$\overline{1}$…
χ	;1$\overline{2497}$…	**:12**	**.1**
Ɛ	;$\overline{1}$…	:$\overline{10\ \chi9}$…	.$\overline{09}$…
10	**;1**	**:10**	.08$\overline{3}$…
11	;$\overline{0\text{Ɛ}}$…	:$\overline{09\ 27\ 83}$…	.$\overline{076923}$…
12	;0$\overline{\chi35186}$…	:08 $\overline{68}$…	.0$\overline{714285}$…
13	;0$\overline{9724}$…	**:08**	.0$\overline{6}$…
14	**;09**	**:07 60**	**.0625**
15	;08579… (M)	:$\overline{07}$…	.05882… (M)
16	**;08**	**:06 80**	.0$\overline{5}$…
18	;0$\overline{7249}$…	**:06**	**.05**
19	;0$\overline{6\chi3518}$…	:05 $\overline{85}$…	.$\overline{047619}$…
1χ	;0$\overline{6}$…	:$\overline{05\ 54\ 65}$…	.0$\overline{45}$…
20	**;06**	**:05**	.041$\overline{6}$…
21	;05915… (M)	**:04 96**	**.04**

Table 5. Digital representation of reciprocals of the dozenal numbers at left in the bases shown in the top row. Terminating digital fractions are shown in bold. Repeating fractions have digital values which are followed by an ellipsis (…). The repeating digits of the digital fractions are denoted by a vinculum placed above the group of digits (e.g. dozenal ;$\overline{2497}$…). Digital fractions with periods longer than six digits are designated "M", meaning that their representations are of maximal length.

The Logarithmic Cycle

Ms. Krieger considered the distribution of divisors on a logarithmic cycle. Figure 7 shows the logarithms of base *r* represented as fractions of a circle, where one full turn = *r*. She asserts the divisors of the base represent the best intermediate scales and division points (like 5¢ or 25¢ coins), the more equally spaced they become, the better they serve as intermediate values.

Bases like 16;, 24;, and ten dozen have relatively uniform logarithmic spacing between their divisors. Other bases like sexagesimal have large gaps between the divisors (cf. 6 and χ in r = 50; in Figure 7) or like 10; or 60;, have relatively close divisor spacings (3 and 4 or 8 and 9). In base 16;, the number 6 and the golden ratio ($\phi = \pm 1;75$): $6^{\phi} \approx \phi^{6} \approx 16;$.

Bases like 24; and χ0; are 2-perfect and 3-perfect numbers. One can devise weights like ⅛, ¼, ½, 1, 2, 4, 8 to give a set of near binary weights that work well with the base. With 2-perfect numbers (6, 24;), the divisor sets {1, 2, 3, 6} and {1, 2, 4, 7, 12;, 24;} add up to double the base. Ten dozen's divisors total up to three times the base.

The 14; divisors of χ0; are scattered over points that are roughly $\chi0;^{(n/19;)}$. Dozenal divisors fall near the square root (at the "six o'clock" position in the diagrams of Figure 7), while those of 20; fall near the cube roots (at the "four and eight o'clock" positions).

Structure and Nomenclature for Twelfty

Twelfty is noted much like Babylonian sexagesimal notation, except that one uses the Hindu-Arabic numerals 0 through 9 for the unit-figures, and a dozenal range of numerals, beginning with 0 through 9, for the decade-figures. Her own notation for ten decades is "V", and is called "teenty" to avoid confusion with the word "twenty". She uses "E" to signify eleven decades, calling it "elefty" to avoid mixing up withthe word "seventy". (In this article, we'll retain the DSA standard dozenal numerals as these are already familiar to the reader.) In her notation, one may omit a preceding decade figure for digits larger than 1. However, zeros must appear in medial positions (such as 1/13; = :08, or in a number like 20 00 05 = decimal 288,005) to avoid confusion, and in the unit-figures of even-decade digits like :60 = ½ and 10 00 = decimal 1200.

Wendy's preference would be to regard the decade- and unit-figures as proper digits

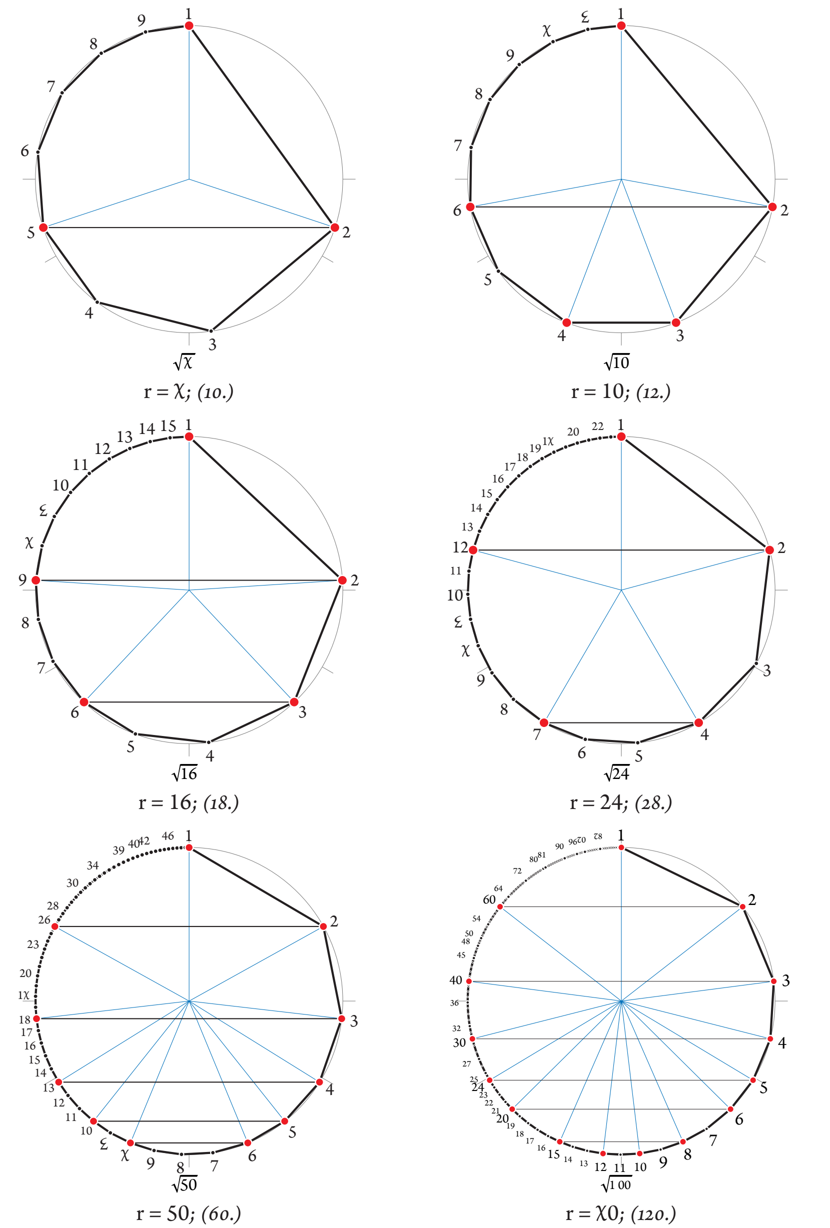


Figure 7. Diagrams of the logarithmic cycles for certain positive integers r. The logarithms of the digits of base r are plotted with zero at the "12 o'clock" position, then moving clockwise toward the logarithm of 2 in base r (log 2/log r), till we reach the the logarithm of r in base r, which equals 1, here represented by a full rotation. Blue lines accentuate the location of the divisors of r, with the trivial divisors occupying the same location at "12 o'clock". Pairs of "reciprocal divisors" lie horizontally from one another. All figures in the diagrams are dozenal, except the diagram for r = χ0; is annotated in "twelfty". These diagrams were produced by data yielded by the Wolfram Mathematica formula Column[Table[List[i–1*N[360*Log[*r*]]], {i, 1, *r*}]], *where r is the positive integer under examination. These diagrams were produced mid 2009 for a separate study by Michael De Vlieger, and extended to illustrate r = χ0;.*

	TWELFTY	DECIMAL	DOZENAL	UPPER STAVES	
one	1:	1.	1;	1□	-teen
ten	10:	10.	Χ;	2□	twenty-
hundred	1 00:	120.	Χ0;	3□	thirty-
ten hundred	10 00:	1200.	840;	⋮	
thousand	1 00 00:	14,400.	8,400;	9□	ninety-
ten thousand	10 00 00:	144,000.	6Ɛ,400;	Χ□	teenty-
cention	1 00 00 00:	1,728,000.	6Ɛ4,000;	Ɛ□	elefty-
ten cention	10 00 00 00:	17,280,000.	5,954,000;	*Lower staffs have the same names as the decimal digits 0-9.*	
million	1 00 00 00 00:	207,360,000.	59,540,000;		
ten million	10 00 00 00 00:	2,073,600,000.	49Χ,540,000;		

Table 6. Krieger's "twelfty" nomenclature and conversions.

of an alternating base rather than components of single digits of a uniform base-ten-dozen. This is a crucial strength of Krieger's "twelfty" system; this enables computation using ordinary decimal and dozenal addition and multiplication rather than the massive uniform base-ten-dozen multiplication tables. Wendy calls a figure, one of two such that compose a single digit, a *staff*. Thus we can call the decade-figure an "upper staff" or a "decade-staff", and the unit-figure as a "lower" or "unit-staff". Each place or *twistaff* of base-ten-dozen are thus composed of a pair of higher and lower staves, in that order. Consider the twelfty number 5 28: (decimal 628.) The upper staff 2□ followed by the lower staff □8 represents the decimal number 28. However, pairing the staves in the opposite way (lower staff of a higher digit and the upper staff of the lower digit) is also important. The lower staff □5 in the twelfty "hundred" position followed by the upper staff 2□ of the lower represents the decimal number 620. We'll call this arrangement an "interstitial pair". Ms. Krieger notes that an interstitial pair with the staves separated by a unit point (e.g. □1:6□ = 1½) replicates the ancient use of mixed bases, decimal to represent multiples, dozenal to represent fractions.

Ms. Krieger asserts that, since twelfty can be thought of as twelve decades, and provided one is familiar with dozenths and decimals, addition in this base is straightforward. Simply add the lower staves using decimal rules, carrying after summing more than ten, and add the upper staves using dozenal rules, carrying after summing more than one dozen:

1 1 1 1 1		1 1 1		09¹⁹ 1	
29 71:48	3551.4	1 87 43 24:	2,985,984	1 ~~10~~ ~~Χ~~0:64	15700 8/15
+ 14 48:72	1728.6	69 53 40:	1,000,000	−75 Ɛ5:40	9115 ⅓
44 00:00	5280.0	+ 4 32:	512	54 Χ5:24	6585 ⅕
		2 36 Χ0 96:	3,986,496		

Wendy uses the "reciprocal divisor pairs" of twelfty such as {2 × 60.} or {10. × 12.} to facilitate multiplication and division (see Table 7). She takes advantage of the fact that a given multiplication problem can be divided into two operations, thus avoiding use of an unwieldy digit as a multiplier. One tool is the "twelve-shift", wherein a multiplier which is a multiple of twelve, say (Χ8:)—decimal 108.—is split into (9:)(12:). The best illustration of this occurs in an example. Let's multiply 5 28: by 84:. We can regard the multiplier (84:) as (7:)(12:). Dealing with the (12:) first, we can use a twelve-shift. The twelve-shift involves dividing the first number into interstitial staff pairs: □5:2□, □8:0□. Using Table 8, □5:2□-hundred becomes 62: hundred, while □8: becomes 96:. Adding

1:	×	1 00:
2:	×	60:
3:	×	40:
4:	×	30:
5:	×	24:
6:	×	20:
8:	×	15:
10:	×	**12:**

Table 7. The reciprocal divisor pairs for twelfty.

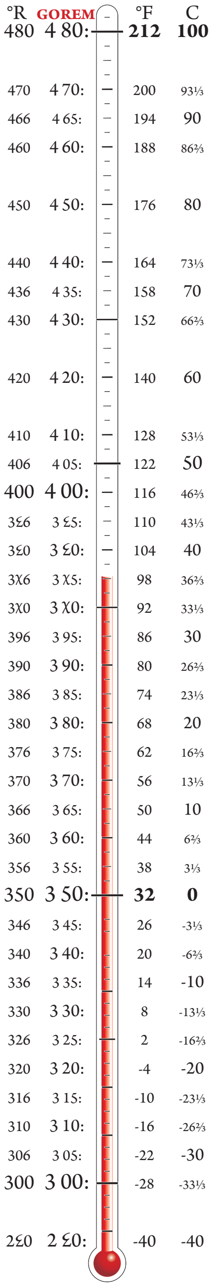
1. HIGHER UNIT-STAFF \ 2. LOWER DECADE-STAFF	:0□	:1□	:2□	:3□	:4□	:5□	:6□	:7□	:8□	:9□	:Χ□	:Ɛ□
□0:	00:	01:	02:	03:	04:	05:	06:	07:	08:	09:	10:	11:
□1:	12:	13:	14:	15:	16:	17:	18:	19:	20:	21:	22:	23:
□2:	24:	25:	26:	27:	28:	29:	30:	31:	32:	33:	34:	35:
□3:	36:	37:	38:	39:	40:	41:	42:	43:	44:	45:	46:	47:
□4:	48:	49:	50:	51:	52:	53:	54:	55:	56:	57:	58:	59:
□5:	60:	61:	62:	63:	64:	65:	66:	67:	68:	69:	70:	71:
□6:	72:	73:	74:	75:	76:	77:	78:	79:	80:	81:	82:	83:
□7:	84:	85:	86:	87:	88:	89:	90:	91:	92:	93:	94:	95:
□8:	96:	97:	98:	99:	Χ0:	Χ1:	Χ2:	Χ3:	Χ4:	Χ5:	Χ6:	Χ7:
□9:	Χ8:	Χ9:	Ɛ0:	Ɛ1:	Ɛ2:	Ɛ3:	Ɛ4:	Ɛ5:	Ɛ6:	Ɛ7:	Ɛ8:	Ɛ9:

Table 8 (above, right). "Reversal" Table. To multiply by one dozen in twelfty, one uses the "twelve shift", which is the first step in most twelfty multiplication problems. Extract the unit-staff of the higher twistaff (the higher unit-figure), then the decade-staff of the lower twistaff (the lower decade-figure) to get an interstitial pair. For example, if we are performing a twelve shift on the number 4 89:15, we would extract the pairs □4:8□, □9:1□, and □5:0□. Using □4:8□, find □4 in the leftmost column, then 8□ in the topmost row. The twelve shift is 56: For the next twistaves, we obtain Χ9: and 60:. Essentially, all we've done is read the high unit-staff as a dozen, and the low decade-staff as a unit. The interstitial pair □4:8□, if interpreted as the dozenal number 48; or four dozen eight, has the decimal equivalent of 56. The interstitial pair □9:1□, interpreted as nine dozen one, is 109 decimally, however, the decade-staff of the result cannot be represented as "10" decades; instead we write "Χ9" (teenty-nine) rather than "109". Thus, 4 89:15 "four hundred eighty nine and an eighth" [decimal 569.125] twelve-shifted is 56 Χ9:60 "fifty six hundred teenty nine and a half" [decimal 6829.5].

these together, we get 62 96:. Multiplying 62 96: by the remaining factor 7: decimally, careful to carry for the lower staves decimally, and for the higher staves dozenally, we find the answer: 3 79 72:. We can use other reciprocal divisor pairs, depending on the multiplier's factors. If we want to multiply the decimal year 2011. = twelfty 16 91: by 40:, we can use its "reciprocal", divide it by 3:, and promote the result one digit (two staves). 16 91: divided by three is 5 40: + 30:40 = 5 70:40. Promoting the result, we obtain 5 70 40: or decimal 80,440. This way, we can avoid multiplication by large digits. Multiplication and division can proceed quite conventionally, so long as one is mindful of converting each higher staff to dozenal notation, i.e., higher staves with a decimal value of ten-decades is written Χ□ (teenty), and eleven-decades as Ɛ□ (elefty). This method is effectively using the {10. × 12.} pair again, this time starting with the factor ten:

89:		89.	89:		89.
× 73:		× 73.	73:)54 17:		73.)6497.
2 27:	[× □3]	267.	−48 80:	[× 8□]	−5840.
+ 51 Ɛ0:	[× 7□]	+ 6230.	5 57:		657.
54 17:		6497.	−5 57:	[× □9]	−657.
			0:		0.

Wendy regards ninths as important numbers, as they turn up often in square roots. She soon learned that 4/9 is :53 40 and 7/9 is :93.40. The very large number of divisors twelfty offers provides useful patterns for writing mixes on bottles, etc. (e.g. 30p *x*, 90p water). If you think this simple household application renders using base-ten-dozen akin to using an atom bomb to snuff out a mosquito, read on. I pressed Wendy for a



more serious application; she provided a sample of higher-space mathematics which we'll describe here in a nutshell.

Ms. Krieger is has studied the regular and semiregular geometric figures of higher space, collectively known as polytopes. These figures are analagous to our three-dimensional Platonic and Kepler-Poinsot solids (the cube, octahedron, the stellated dodecahedra, etc.) and the semiregular figures such as the cuboctahedron and the icosidodecahedron. As an example, she computes aspects of the measurements of figures in 4 and 5 dimensions using square roots. Wendy says she's often on the go, and likes to make use of down time to explore using mathematics, such as time at a cafe or waiting on a bus. Twelfty, with its more compact significands, its many divisors, the short preferred intervals, and her alternating-base notation using staff-pairs, facilitates math on the go. She can boil out the three unique prime factors of ten-dozen (2, 3, and 5) to leave only the primes not included, and can use the short preferred intervals she's familiar with to determine what prime numbers are at play in a given result.

Wendy has written computer algorithms to help analyze the realm of higher space. She's also devised complete weights and measurement systems for twelfty.

Conclusion

Wendy Krieger has been using twelfty for about two and a half dozen years. To her, expressions like "sixty-sixty" have displaced "fifty-fifty". Over the years, through conducting arithmetic and producing a system of weights and measures, Wendy has become fluent in twelfty. She's resolved complex multiplication and square-root algorithms in the base, as well as tables that facilitate other operations. Some common calculations others may have routinely conducted in decimal Wendy claims she's done only using twelfty.

Wendy Y. Krieger is a brilliant intellect, with many more interesting tools and techniques like criss-cross multiplication, quarter-square multiplication, and continued fractions. We could literally write a book about her adventures with the alternating-base version of base ten-dozen she calls "Twelfty".

Editor's notes: *See pages 28; and 29; for notes and references. This interview was conducted over a series of electronic mails. Ms. Krieger provided the Neugebauer and Butterworth references, while the balance of the references and all the citations were provided by M. De Vlieger.*

Visit Wendy Krieger's website: http://os2fan2.com/.

Figure 8. Wendy Krieger's "gorem" scale, compared to Rankine in dozenal (°R), Fahrenheit (°F), and centigrade (C).[27] *In the gorem scale, the "forty below" equivalency point between centigrade and Fahrenheit lies at 2 Ɛ0:. At standard pressure, the freezing point of water lies at 3 50:, its boiling point at 4 80:, and the midpoint of its liquid phase at 4 05:. Between 3 00: and 4 00: gorem lie everyday temperatures. Our freezers and refrigerators cool to around 3 23: and 3 57: gorem. Human body temperature lies around 3 Χ5: gorem. Room temperature is a comfortable 3 80: gorem.*

Notes.

1 Krieger 2002.
2 Krieger, Wendy Y. "PolyGloss", Retrieved January 2011, <http://os2fan2.com/gloss.htm>, sections "Decimal (dec)", "Twelfty (twe)", and applied in a table at entry "heptagon".
3 Krieger, Wendy Y. "Long counting", Retrieved January 2011, <http://os2fan2.com/twelfty.html>, and "Twelfty — long numbers", Retrieved January 2011, <http://os2fan2.com/pttwelfty.html>.
4 Krieger, Wendy Y. "Twelfty — long numbers", Retrieved January 2011, <http://os2fan2.com/pttwelfty.html>, section "The Practical Base".
5 Weisstein, Eric W. "Regular Number", Wolfram MathWorld. Retrieved January 2011, < http://mathworld.wolfram.com/RegularNumber.html >.
6 Consult the *IEEE 754-2008 Standard for Floating Point Arithmetic*, IEEE (formerly the Institute of Electrical and Electronics Engineers), available in January 2011 at <http://ieeexplore.ieee.org/xpl/mostRecentIssue.jsp?punumber=4610933>.
7 Ore 1948, Chapter 5, "The Aliquot Parts", page 86, equation 5–2.
8 Melville 2005, page 1, first paragraph.
9 Melville 2005, page 1, "No absolute scale of the numbers is indicated and so, in effect, we treat as a number and its reciprocal any pair of numbers whose product is a power of 60, and hence denoted by 1 in the sexagesimal system."
Χ Neugebauer 1962, Chapter 2, "Babylonian Mathematics", page 32, table at the top of the page. See also Melville 2005, page 2, table 1, "The standard table of reciprocals".
Ɛ Stephenson 2010, section "The difference between a counting board and an abacus", specifically, "Both the [modern] abacus and the counting board are mechanical aids used for counting; they are not calculators in the sense we use the word today. The person operating the abacus performs calculations in their head and uses the abacus as a physical aid to keep track of the sums, the carrys, etc."
10 Menninger 1969, page 299, "The nature of the counting board: the Salamis Tablet".
11 Stephenson 2010, section "Unused Dashed Lines", specifically, "If all numbers are entered with the most significant digit in the top position [of the lower zone], therefore as a fraction of one, along with the appropriate radix shift, then no complex positioning rules are needed. … The Babylonians [sic] lack of a radix symbol and elimination of complex positioning rules are strong evidence that the top grid on The Salamis Tablet is used for storage and manipulation of a radix shift, what we call an exponent of the base."
12 Stephenson 2010, section "Unused Dashed Lines", specifically, "The Romans were borrowers. They borrowed The Salamis Tablet from the Greeks, but the Greeks borrowed it in turn from the Babylonians." See also section "Designers of the Salamis Tablet".
13 Stephenson 2010, sections "Clues" and "Roman hand abacus".
14 Menninger 1969, pages 299–303, "The nature of the counting board: the Salamis Tablet".
15 Menninger 1969, pages 305–306, "The nature of the counting board: the Roman hand abacus". See also Stephenson 2010.
16 Butterworth 1999, pages 183–196, "Numbers in the brain: 2. Inside the mathematical brain" and "Numbers in the brain: 3. Understanding numbers".
17 Menninger 1969, page 167, "Babylonian influence: Babylonian sexagesimal system".
18 Menninger 1969, pages 163–169, "Babylonian influence: Babylonian sexagesimal system", see also Neugebauer 1962, Chapter 1, "Numbers", pages 14–16, items 11–12.
19 Neugebauer 1962, Chapter 1, "Numbers", page 20, item 14.
1Χ Neugebauer 1962, Chapter 1, "Numbers", page 20, item 14, "in all periods the context alone decides the absolute value of a sexagesimally written number". See also Menninger 1969, pages 164–165, "Babylonian influence: Babylonian sexagesimal system", especially "The requirements of ordinary life in Babylonia, as everywhere else, demanded the use of the common fractions of a measure, in this case ½, ⅓, and ⅔. As time passed it became necessary to express the fractional parts of the larger measure as whole numbers of the smaller, for example, ⅓ mina in shekels, and thus to 'bond' the two separate groups of measures."
1Ɛ Ifrah 1981, Chapter 13, "Mesopotamian numbering after the eclipse of Sumer", "How did Babylonian scientists do their sums?" page 154–156, especially figure 13.68: Twenty-five times table.
20 Neugebauer 1962, Chapter 2, "Babylonian mathematics", page 31–34, item 18.
21 Neugebauer 1962, Chapter 2, "Babylonian mathematics", page 34, item 19.
22
23 Menninger 1969, pages 167–168, "Babylonian influence: Babylonian sexagesimal system".
24 Ore 1948, Chapter 13, "Theory of decimal expansions", page 324.
25 Ore 1948, Chapter 12, "Euler's theorem and its consequences", page 277, theorem 12-4 (Fermat's Little Theorem).
26 Wolfram Mathematica 7.0 computed values to a dozen places: ϕ = 1.74ƐƐ6772802Χ…, ϕ^6 = 15.Ɛ3Ɛ8509941ΧƐ…, and 6^ϕ = 16.1Χ965Ɛ408058…
27 Actual value of freezing point of water at standard pressure is 491.7°F. The dozenal Rankine and "gorem" equivalents in Figure 8 are rounded up 0;37249…°R = 0:36 gorem = 0.3°F. Freezing is thus 3 49:84, boiling 5 69:84, and human body temperature 4 86:36 gorem. See <www.wolframalpha.com>.

problem corner

solution from page 10; *by Gene Zirkel*

Here is the solution to the cryptogram!

THE BEST ARGUMENT FOR BASE
FTQ NQEF MDSGYQZF RAD NMEQ
TWELVE OVER BASE TEN IS A LOOK AT THE FRAC-
FIQXHQ AHQD NMEQ FQZ UE M XAAW MF FTQ RDMO-
TIONAL EXPRESSION FOR 1/3 IN BOTH BASES.
FUAZMX QJBDQEEUAZ RAD 1/3 UZ NAFT NMEQE.

problem
solution in next issue

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The Basics of Systematic Dozenal Nomenclature

Systematic Dozenal Nomenclature is a concise, coherent, and consistently dozenal way to refer to numbers, as well as to form words for arbitrary numbers, intended to be international in scope and comprehensive in nature. A full exposition of this system (SDN) will have to wait for another *Bulletin*; however, a basic overview will be offered here. (Note that this article uses Pitman characters, as is the preference of the author.)

Dozenalists have long argued about the proper way to speak when using dozenals. Is it "ten" or "dek?" Is it "eleven," "el," "elv," "elf," or something else entirely? Do we say "twelve," "dozen," or what? And just what do we call a dozen gross, anyway?

Traditional English has a system for talking about dozenals, simple enough, which goes like this:

Number (doz.)	Word
10	dozen
100	gross
1000	great-gross

We can extrapolate from these three; for example, the number "7432↊9" is "seven gross four dozen three great gross, two gross ten dozen nine." This works, in the sense that it is functional; and this is usually the very best way to inroduce dozenals to someone who's not familiar with them. However, it is clunky, verbose, and (worst of all) ugly. Something better is needed if we are to make dozenals mainstream.

For many years, the DSA used the "do-gro-mo" nomenclature, which is essentially a shorthand for the above "plain English" system. "Dozen" becomes "do," "gross" becomes "gro," and "great gross" becomes "mo," forming the following system well-known to anyone who has perused the older issues of our *Bulletin*:

10	Do	0.1	Edo
100	Gro	0.01	Egro
1000	Mo	0.001	Emo
1 0000	Do-mo	0.0001	Edo-mo
10 0000	Gro-mo	0.0000 1	Egro-mo
100 0000	Bi-mo	0.0000 01	Ebi-mo
1000 0000	Tri-mo	0.0000 001	Etri-mo

And so forth, just as far as one cares to take it. This is a much more robust system that the "plain English" method, and served us well for many years. Some of us continue to use it.

However, even this system has a flaw: it's too provincial. That is, it fits in pretty well with English (though the fractional names do sound a bit funny), but it fits in poorly

in other languages. Dozenal isn't meant to be limited to the Anglosphere, and never was; it's *mathematically*, not *culturally*, the superior base. So as well as this system has worked, something better is still needed.

Enter Systematic Dozenal Nomenclature (SDN). A group of dozenalists on the DozensOnline forum, ably led by John Kodegadulo, has put together a system which is comprehensive; systematic; simple; and international. The system manages all these traits at once by adopting our elder, Ralph Beard's, Principle of Least Change: it uses things which are familiar to all of us, changes them as little as possible to accomplish our goals, and then runs with them.

The familiar basis for SDN is the set of numerical particles used by the International Union for Pure and Applied Chemistry (IUPAC). IUPAC takes very familiar Latin and Greek roots, most of which we already know, to form the names of new chemical elements until they can be given their "official" names; SDN adds two new particles for ten and eleven, and then applies them in predictable, easy ways.

Further particles can be formed using the familiar principles of place notation; e.g., "25" is simply "bipent," the number words being put together just as the number digits are.

Finally, we need something to indicate exponentiation. So we add to these the suffix "qua" if the power is positive, and "cia" if the power is negative.

Num.	Part.	Pos. Power	Neg. Power
0	Nil	Nilqua	Nilcia
1	Un	Unqua	Uncia
2	Bi	Biqua	Bicia
3	Tri	Triqua	Tricia
4	Quad	Quadqua	Quadcia
5	Pent	Pentqua	Pentcia
6	Hex	Hexqua	Hexcia
7	Sept	Septqua	Septcia
8	Oct	Octqua	Octcia
9	Enn	Ennqua	Enncia
↊	Dec	Decqua	Deccia
↋	Lev	Levqua	Levcia

So "10" is now "unqua" (10^1); 100 is "biqua" (10^2); 1000 is "triqua" (10^3); 1 0000 is "quadqua" (10^4); and so on. We literally simply count the digits after the first, use the corresponding word, and call it a day.

For example:

7 82↋4 ↊↋45

In the "plain English" system, this number is borderline impossible to speak; that is, it could probably be done, but the result would be so unwieldy that speaking it would communicate practically nothing to the listener. In the *do gro mo* system, we get the rather manageable:

Seven gro bi-mo eight two el four dek el four five

But to speak this, we must recall that "bi-mo" means "10^6" (though it has nothing indicating "6" in it), we must count out the digits, we must see that we've got seven gro of bi-mo, and then voice the number accordingly. Doable, certainly; but excessively complex as well as too provincial in its roots.

In SDN, we simply count the digits after the first (eight), select the corresponding and very familiar numerical particle (oct), add "qua" to it (since it's clearly a large number rather than a small one), and we've got the answer:

Seven octqua eight two el four ten elv four five

SDN is at once simpler, more comprehensive, and more international than other systems.

Of course, there is no need to give up traditional ways of speaking; one may as well attempt to prevent metric countries from selling their meat in half-kilogram "pounds." SDN *can* replace native modes of speaking in dozens (your author has used it such for some time), but it need not. It does, however, provide a way of speaking in dozens which is acceptable to the whole of the international community, while at the same time being thoroughly dozenal in its principles and its intent.

The primary inventor of SDN, John Kodegadulo, has prepared a full exposition of the power of the system, which space unfortunately did not permit for this issue; but look forward to that exposition in the next *Duodecimal Bulletin*. ⁝⁝⁝

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