COUNTING IN DOZENS

1	2	3	4	5	6	7	8	9	X	Ɛ	10
one	two	three	four	five	six	seven	eight	nine	dek	el	do

Our common number system is decimal - based on ten. The dozen system uses twelve as the base, which is written *10*, and is called *do*, for dozen. The quantity *one gross* is written *100*, and is called *gro*. *1000* is called *mo*, representing the meg-gross, or great-gross.

In our customary counting, the places in our numbers represent successive powers of ten; that is, in 365, the 5 applies to units, the 6 applies to tens, and the 3 applies to tens-of-tens, or hundreds. Place value is even more important in dozenal counting. For example, *265* represents *5* units, *6* dozen, and *2* dozen-dozen, or gross. This number would be called *2 gro 6 do 5*, and by a coincidence, represents the same quantity normally expressed as 365.

Place value is the whole key to dozenal arithmetic. Observe the following additions, remembering that we add up to a dozen before carrying one.

94	136	Five ft. nine in.	5.9'
31	694	Three ft. two in.	3.2'
96	3Ɛ2	Two ft. eight in.	2.8'
19Ɛ	1000	Eleven ft. seven in.	Ɛ.7'

You will not have to learn the dozenal multiplication tables since you already know the 12-times table. Mentally convert the quantities into dozens, and set them down. For example, 7 times 9 is 63, *which is* 5 dozen and 3; so set down *53*. Using this "*which is*" step, you will be able to multiply and divide dozenal numbers without referring to the dozenal multiplication table.

Conversion of small quantities is obvious. By simple inspection, if you are 35 years old, dozenally you are only *2Ɛ*, which is two dozen and eleven. For larger numbers, keep dividing by 12, and the successive remainders are the desired dozenal numbers.

```
12 ) 365
 12 ) 30 + 5
  12 ) 2 + 6
       0 + 2    Answer: 265
```

Dozenal numbers may be converted to decimal numbers by setting down the units figure, adding to it 12 times the second figure, plus 12^2 (or 144) times the third figure, plus 12^3 (or 1728) times the fourth figure, and so on as far as needed. Or, to use a method corresponding to the illustration, keep dividing by X, and the successive remainders are the desired decimal number.

Fractions may be similarly converted by using successive multiplications, instead of divisions, by 12 or X.

Numerical Progression

1	One		
10	Do	.1	Edo
100	Gro	.01	Egro
1,000	Mo	.001	Emo
10,000	Do-mo	.000,1	Edo-mo
100,000	Gro-mo	.000,01	Egro-mo
1,000,000	Bi-mo	.000,001	Ebi-mo
1,000,000,000	Tri-mo	and so on.	

Multiplication Table

1	2	3	4	5	6	7	8	9	X	Ɛ
2	4	6	8	X	10	12	14	16	18	1X
3	6	9	10	13	16	19	20	23	26	29
4	8	10	14	18	20	24	28	30	34	38
5	X	13	18	21	26	2Ɛ	34	39	42	47
6	10	16	20	26	30	36	40	46	50	56
7	12	19	24	2Ɛ	36	41	48	53	5X	65
8	14	20	28	34	40	48	54	60	68	74
9	16	23	30	39	46	53	60	69	76	83
X	18	26	34	42	50	5X	68	76	84	92
Ɛ	1X	29	38	47	56	65	74	83	92	X1

The Duodecimal Bulletin

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March - 1954

THE DUODECIMAL SOCIETY OF AMERICA
20 Carlton Place ~ ~ ~ ~ ~ Staten Island 4, N. Y.

THE DUODECIMAL SOCIETY OF AMERICA

is a voluntary nonprofit organization for the conduct of research and education of the public in the use of Base Twelve in numeration, mathematics, weights and measures, and other branches of pure and applied science.

Full membership with voting privileges requires the passing of elementary tests in the performance of twelve-base arithmetic. The lessons and examinations are free to those whose entrance applications are accepted. Remittance of $6, covering initiation fee ($3) and one year's dues ($3), must accompany applications.

The Duodecimal Bulletin is the official publication of the Duodecimal Society of America, Inc., 20 Carlton Place, Staten Island 4, New York. F. Emerson Andrews, Chairman of the Board of Directors. Ralph H. Beard, President. George S. Terry, Editor. Separate subscriptions $2.00 a year, 50¢ a copy.

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The Duodecimal Bulletin

All figures in italics are duodecimal.

THE ANNUAL MEETING

The Annual Meeting of the Duodecimal Society of America was held at the Gramercy Park Hotel, New York City, on Thursday, 21 January 1954. President Beard called the meeting to order at 2030 EST., and advised that it had been thought advisable this year to restrict the meeting to the membership only, in order that there might be open discussion of the plans and prospects of the Society.

Our work in familiarizing the public with the use and application of the duodecimal base continues at about its usual pace. There has been only one issue of the Duodecimal Bulletin, but we have distributed, in addition, about 1800 sets of our literature. Among these were bulk orders for the annual convention of the National Council of Teachers of Mathematics at Detroit, and for the convention of their Florida Council.

In continuation of his work with the encyclopedias, Mr. Andrews arranged for an article on duodecimals for the Encyclopedia Americana, and received expressions of interest in similar material for future revisions of the Encyclopedia Britannica and the Columbia Encyclopedia.

Three new members have been enrolled, but, since we have lost touch with one or two, the membership stands at 75. Some 400 names have been dropped from our mailing list, through a check by return postcard, and the list now includes about 1000 names.

The U.S. Department of Education is listing Mr. Andrews' "An Excursion in Numbers," in its catalog of available aids to education, and a similar listing by the National Council of Teachers of Mathematics has been approved.

Treasurer Humphrey presented a very satisfactory report of the Society's financial condition. Increased mailing costs for the year were offset by there being only one issue of The Bulletin. Receipts were $1325.80, of which $301.16 was from initiation fees and dues, and contributions amounted to $1024.64. Expenses of $1430.30 exceeded receipts by $104.50. The cash account was replenished by the sale of a bond from the Endowment Fund, (which now amounts to $3000) and the balance carried forward is $1062.48.

In expressing its thanks to Mr. Humphrey for his fine report the Society recorded its deep gratitude to those generous members who make our work possible.

Mr. Stevens reporting for the Nominating Committee, recommended that

F. Emerson Andrews
William Shaw Crosby
and Louis Paul d'Autremont

of the Class of 1954 of the Board of Directors be re-elected as the Class of 1957, and that the Nominating Committee be continued for another year without change. These elections were carried.

Mr. Beard announced that the Board of Directors had, that afternoon, re-elected the same staff of officers for the ensuing year.

Kingsland Camp displayed his new planisphere, finished in fine, heavy plastic and arranged for use in any latitude, with the constellations and those stars brighter than the 5th magnitude set forth with beautiful clarity. These are now available at $10. each.

Edward Pharo offered to arrange for the translation of An Excursion in Numbers into Esperanto, as he believes that those interested in this universal language would also be interested in the base best suited for universal use in numbers and measures.

With the serving of refreshments, the meeting dissolved into the warm discussion and interchange of ideas that is so much enjoyed.

ORIGIN OF THE FOOT AND INCH

by Warren H. Chapin

The language of most peoples contains a distinctive word for twenty, such as our score, and also special words for "half" and "quarter", which are aliquot parts of twenty, as if in prehistoric times when language was being formed, these were specially important numbers and fractions.

The first measure of length in history was the cubit, nominally the length of the forearm, from elbow to end of middle finger, but since this length would vary even among adult men, the Egyptians standardized their "Royal Cubit" at 20.62 of our inches, as shown most accurately in the Great Pyramid (Ency. Brit. Ancient Measures).

A nation of individuals, most of whom had no rules or pencils, or pockets to keep them in, could carry this standard around with them by checking in the marketplace, how many fingerbreadths to add to a certain fingertip to equal this standard.

In my case it takes four fingerbreadths added to length from elbow to tip of forefinger to equal above cubit, although my forearm is shorter than average. By laying four fingers down to start, butting elbow against them, I can measure off a length of 123¾" (6 cubits) with an error of less than ¼". Probably with everyday use many would become quite proficient at this.

Probably the above cubit was first dubdivided into twenty parts, each equaling 1.03 of our inches.

Later it was found that 12 of these inches made a convenient unit of length, which was widely spread over Europe and the Middle East, by travel and trade, as far away as Stonehenge, and was used in mediaeval England.

Since the Babylonian cubit was in general longer than the cubit of the Great Pyramid, the foot first transmitted by them throughout Europe by the Phoenician navigators, was longer than 12/20 of 20.62.

I was also interested in what the Mayas used for a cubit or foot. The only reference I can find is in these articles discussing the 20.62 cubit, which states that this same cubit of 20.68 appears in stone buildings of New Mexico. Did the ancient Mayas, Aztecs and Cliff Dwellers of America independently arrive at practically the same primary unit of length?

About the same time, and also of Egyptian or Babylonian origin, was a unit based on ½ of the diagonal of the above inch called the digit. The ancient Greeks used 16 of these digits to make their foot of 11.66" as used in the Parthenon. This, when transmitted to Rome and subdivided into 12 parts, became the Roman foot of 12 unciae or inches, and was widely spread by them to England, etc.

Apparently the Greeks, and from them the Romans, selected the cubit of the Great Pyramid of 20.62 of our inches as the starting point to calculate their standards of length, because of the accuracy, durability and uniqueness of its construction.

The Greeks, using a system of halving and doubling or duplation in their arithmetic, preferred a foot subdivided into 16 parts or digits, a digit being 1/40 of the diagonal of the 20.62 cubit (.729"), the later Greek foot of the Parthenon being 16 of these digits or 11.66 of our inches.

This same foot of 11.66 was taken to Rome and finally divided duodecimally, and appears in old tables of Latin and Scandinavian countries.

If one takes the average of the two above standard feet, a surprisingly close value is obtained for the present English foot, thus:

12.37 = 12/20 of 20.62" cubit.
<u>11.66</u> = 16/40 of diagonal of 20.62" cubit.
Average 12.01"= present English foot very nearly.

Our ancesters, as if to avoid the inconvenience of having two different lengths of feet, averaged them to a single standard.

How did these standards get to England with their close theoretical relation to the 20.62 cubit? I conjecture that Boethius, called by some the last learned Roman, who wrote a treatise of elementary arithmetic widely used as a text in the Middle Ages, might have known the origin of the different feet in use in the various Roman provinces, and suggested the compromise which resulted in the present English foot. [1,2,3]

English or Saxon measures		Drusus or Belgic measures	
12 inches	1 foot	20 inches	1 cubit
36 inches	1 yard	40 inches	1 ell or double cubit
5½ yards	1 rod	10 cubits	1 rod
		1/3 ell	1 Belgit foot

Since the time of the Sumerians, land has been measured in units of 10 cubits or 1 rod.

Drusus, Roman conqueror of what is now Belgium and Holland, could not get the natives to adopt the Roman measure for land. They preferred a system based on the above table, on which our land measures are based, with slight modification.

Alfred the Great studied in Rome for several years. Later, he, himself, translated into Saxon some of the works of Boethius.[1,2,3]

King Edgar (Alfred's great-grandson) decreed that the measures at Winchester (Saxon capital) should be the standard.[3]

In the time of William the Conqueror, these Saxon standards were taken unchanged to Westminster.[3]

The length of the English foot has not changed appreciably in several centuries.[1]

If the above is substantially correct, it shows how tenaciously the human race has clung to the inch throughout history; for the reason, I think, that the size of the inch is such that when working on any single piece of material by unaided eyesight, the bulk of all problems both in multiples and fractions comes within the multiplication and addition tables.

From this standpoint the centimeter (Metric barleycorn) is just right for a race of midgets with correspondingly finer eyesight.

References:

1 Ency. Brit. 11th Edition
2 Ency. Brit. present edition
3 International Encyclopedia
See also, History of Mathematics, Cajori, Life of Boethius.
Mathematics, Magic and Mystery, Bakst, p 102.
The Great Pyramid Mystery.

THE CONVERSION OF ENIAC PI

by Ralph H. Beard

The computation of Pi to 2035 decimal places, and the computation of e to 2500 decimal places on the ENIAC at Aberdeen has been reported in M.T.A.C., v. 4, no 29, Jan. 1950. A review of that report was published in the Duodecimal Bulletin, v. 6, n. 2, Aug. 1950.

In examining the value published in the report, we were excited to discover that the decimal very nearly terminated at the 761st place. The figures for places 761 to 770 are as follows:

761			764			767			770
4	9	9	9	9	9	9	8	3	7

and the rounding of Pi as 5 in the 761st place gives an excellent value. But we became increasingly curious as to whether any similar phenomenon would occur in the duodecimal value.

We were appalled by the prospect of the labor involved in converting 761 places of decimals to duodecimals, but our curiosity would not allow us any peace. And the mechanics of the job started to become interesting. We have a 10-bank decimal Friden computer, and a test run was made on the conversion of 78 places, using blocks of 9 figures and a constant multiplier of 12^9. An awkward overlapping developed in the 19th figure of the products, but it was found that a constant multiplier of 12^8 would yield 18-figure products without overlap. It also became clear that to eliminate error it would be necessary to perform each step of the conversion twice and carefully compare the two separate results. In completing the multiplication of each step, 9 decimal figures would be extruded, and these had to be manually converted to 8 duodecimal figures by successive division by 12.

Upon the completion of the conversion of the rounded value of π for *535*(761) places, the results were forwarded to George S. Terry for review. Mr. Terry commented that though the error in rounding was remarkably small, the conversion process for so many places would magnify it surprisingly. His clear exposition showed that rounding the 761st place to 5 involved an error of $.0^7$ 163 (or .000 000 016 3) in that place, which was an error or $.0^{767}$ 163 in the first place. This minute error, multiplied by twelve 761 times, (that is, by 12^{761}) would produce 55 more whole number places than zeros in the product.

Consequently, it was decided to extend the conversion sufficiently to drive *535*(761) good places. This required an extension to *609*(873) terms for a reason peculiar to the plan of work used.

With 12^8 as the constant multiplier, there was an extrusion of 8 duodecimal figures at each step. But, since 9 figures were put into the Friden at each multiplication, the terminal cut-back consisted of 9 figures at each step. While this had the disadvantage of added work, it was more than offset by the increased dependability of the results, as being definitely free from possible modification by further terms.

The practice of doing the multiplication of an entire line of figures, then repeating the work again on separate work-sheets, and carefully comparing the two results, seems to be reliable. The supplemental results for the extended computation were carried through an overlap of 9 to 36 figures of the original work for reliable blending, - without the discovery of any error.

The completed conversion now includes *546*(774) good places, and it seems reasonably safe to publish the results for *540*(768) places, not as the conversion of a rounded figure, - but as the conversion of 768 places of the Eniac Pi.

It might be well to state that these conversion figures, while deemed reliable, cannot be regarded as positively accurate until both the Eniac computation, and this conversion of those decimal figures, have been confirmed by an independent repetition of this work. Our results follow.

3.184809493Ɛ91		
8664573X6211	*X71626Ɛ8X546*	
ƐƐ151551X057	*87621849Ɛ849*	*253X88X50X43*
29290X7809X4	*X8225616Ɛ442*	*ƐX0944572315*
92742140X60X	*796X31737Ɛ22*	*X387707Ɛ01X3*
55256X0661X0	*9Ɛ2391489853*	*92982210Ɛ748*
3753X3XX5480	*943Ɛ87637256*	*095207139064*
5646880181X3	*16447236Ɛ027*	*630797279495*
683083272ƐƐƐ	*X421XX17X38Ɛ*	*57X92Ɛ8868Ɛ7*
X0X370Ɛ12265	*52X18X838Ɛ01*	*73Ɛ3165X88ƐƐ*
529X828903Ɛ4	*514X51144X23*	*63406ƐƐ57Ɛ0Ɛ*
Ɛ256Ɛ8403759	*315X3009X890*	*0995335Ɛ507Ɛ*
	6Ɛ61Ɛ8Ɛ48X62	*691845310018*
		968X98161232

56Ɛ6X89Ɛ20X0		
086X74620ƐƐX	*346467048191*	
6Ɛ8618340608	*9244X3500252*	*283150564Ɛ85*
X041Ɛ99X6Ɛ25	*613891989625*	*57X3277X8Ɛ43*
10X789834843	*949802Ɛ10856*	*1684X022899X*
Ɛ8X0X9965729	*0816Ɛ85966X9*	*59348041X2X3*
108651Ɛ26914	*Ɛ0632X8ƐX7X9*	
382Ɛ74996918	*X22Ɛ0958Ɛ1Ɛ3*	
X80444899416	*2146Ɛ8650223*	
454957015Ɛ22	*077XX2Ɛ263X1*	
124242627444	*71888X826165*	
27091326X851	*394274ƐX9Ɛ67*	
	Ɛ34X9X653437	

While we were in the swing of the conversion process, it was easy to do a brief conversion of Eniac e . So we added a run of *139*(189) decimal places, which yielded a conversion of *120*(168) good duodecimal places, as follows:

2.875236069821
9ƐX71971009Ɛ
388XX8766760
256427278 6ƐƐ
923Ɛ31032566
054257348716
7Ɛ0008877450
014231ƐƐ98Ɛ9
06X47X4350Ɛ5
453024X21814
12979075ƐX5Ɛ
42X096668813

396431806 33X
8Ɛ5442798843

While nothing developed in either of these conversions that was as exciting as the 761st decimal place of Pi, we did find good usable abbreviations of Pi at places *4*, *20*, and *99*, - and for e at places *6*, *18 3X*, and *62*.

SO YOU WANT TO REFORM WEIGHTS AND MEASURES

by Paul Adams

Early essayists used to begin their endeavors by directing them to the "Gentle Reader". We may suppose that the writer in those days had some excuse for imagining that some real counterpart of the "gentle reader" existed to receive his efforts. At no time in history, however, could a writer on the subject of weights and measures reform have nourished a sustained belief in such an unreal audience. He would be far closer to the truth if he headed his work as being for the benefits of "Fellow Fanatics" or "Brother Bigots". Such an opening, apart from being more honest, would make the inevitable hail of brickbats and invectives come as less of a surprise to the perhaps well-intentioned author, and it might also serve to put the usually defenseless reader on his guard.

One of the striking aspects of the literature of weights and measures is the authoritarian vein in which it is almost all conceived and written. At first, one is inclined to believe that this is a field in which the amateur has no proper place, and it is only later, if one carries one's interest this far, that one begins to suspect that it is a field that only amateurs dare to enter. It might not even be too extreme to suggest that the complex nature of the subject is not likely to allow for any experts at all.

Before concluding my introductory remarks, it would indeed be in keeping with the spirit of the times to confess to past bigotry and biasedness about my subject -- but -- I have now seen the light and become a chastened and reformed character.

In the case of the sophisticated reader who doubts that the admission of sin can really be of much saving grace in such a subject, I can only console him with the further admission that I am with him all the way.

The original intention of this article was to illuminate some of the hidden hazards that almost invariably trap the unwary in their approach to metrological reform. Being trapped in this kind of sin doesn't seem bad at all to the sinner for he has plenty of company even if he doesn't agree with any of it. The worst that can happen to him is that in an after-life, he is compelled to live in a purgatory where each sinner must use his own system of weights and measures, Hell being reserved for those whose basic standards are to each other as infinitely repeating decimals... I make this theological point in hopes of striking terror into the hearts of those for whom there is still some hope.

One of the questions which always seem to plague the layman when he happens to stumble upon some reform or other is "Why?" It is a serious mistake for the enthusiastic reformer to enter into technicalities. The layman, while he might disclain the relationship, has a strong kinship with the metaphysician. He requires an answer that uplifts the soul and does away with his curiosity. In a pinch one can do away with the bit about "uplifting" but by no means leave our layman with any trace of curiosity. He would find this failure on our part unforgivable and hasten to consult other more weighty oracles. (I beg your pardon!)

The solution to this dilemma is to give our layman a demonstration of the perfection that our reform achieves in giving the exact number of angels that can dance on the head of a pin as the perfect square of our basic linear unit. The consideration of the "why" of our reform is the beginning of salvation for the weights and measures sinner. This ordinarily requires considerable time.

Now time measure is silly in its very conception if we don't find a fundamental relation between events that concern human beings. This problem may be stated in a nutshell as finding the common need of a husband figuring out how long his wife will take in getting ready to go out and an atomic physicist in predicting the birth of twin neutrons. A unit of time measure based on this common need will give a least common multiple of history, allowing us to figure exactly, modified by π , of course, the time that must elapse between the Battle of Marathon and the very last world war fought by man. So much for time!

The nomenclature we are to use is a problem of real seriousness. Two schools of thought exist. There is the classic school which believes in returning to ancient languages for our unit names and relations, and the modern school which is sub-divided into advocates of slang and of theoretical phonetics. Fundamental to all schools, however, are monotonousness, lack of intelligibility and slavish conformity.

Favorers of weights and measures change must not neglect the legal aspects of their problem. No method to advance your cause from this angle should be overlooked. Hardest to accomplish is getting fellow exponents of your metrology into key positions in the legislative branches of government, or better still, into some governmental branch dealing directly with existing standards. Once in one of these vital spots, a judicious interpretation of legal terminology or an ingenious twist to a definition will gradually permit you to add to the already unpopular inefficiencies of the existing system. When you begin to detect signs of social unrest, you are then in an incomparable position of advantage from which to present your own reform.

Another legal goal for which we may strive is the compulsory use by a country of our weights and measures. "Compulsion". This is indeed a magic word and by saying it aloud to oneself, yoga fashion, visions of "unity" arise before one. After all, you say, only ignorance and apathy keep the people from adopting the improvements we propose. What then is more logical than to promote excellence by making the use of inferior measures both expensive and painful. Jail is really too good for the wretches who resist improvement.

Of course, as virtue is not always immediately recognized, we many find ourselves persecuted by some unscrupulous group of self-seekers who are using this method to gain arbitrary power. Should this situation develop, one must go immediately underground and keep the fire of freedom burning by smuggling in forbidden standards and ratios to be used even under the tyrant's eye. These contraband units, being smuggled in duty-free, are sold at a fancy profit. These profits, invested wisely, provide a comfortable stake for use on liberation day.

It is obvious that the weights and measures reformer lives essentially for the realization of three high purposes: 1. Unity 2. Naturalness and 3. Conformity to Number.

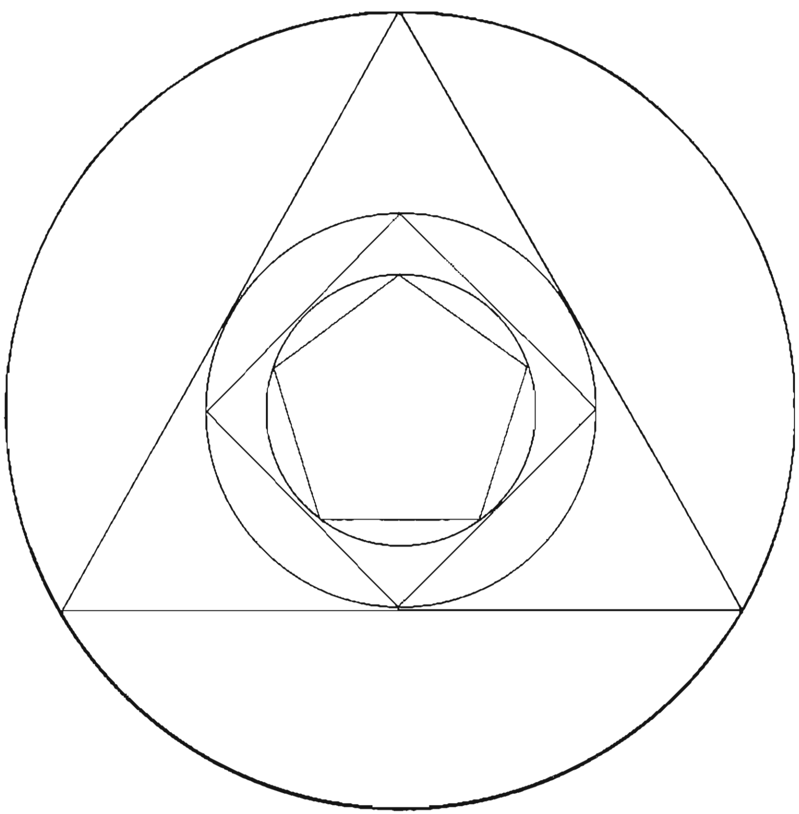
The ideal of Unity implies that one system of measures must surplant all other systems throughout the world. This awesome spectacle of everyone from Hottentots to Esquimaux using one set of measures is marred even in theory by the small detail that there is more than one reformer set upon realizing this aim. It is not hard to imagine, however, the force this high motive lends to the zeal of our lovers of efficiency, and we should not be surprised by the utter disregard of fact and the unethical behavior this noble vision excuses.

Aim Number Two deals with the setting up of physical standards based upon natural phenomena so that these standards can be reproduced or verified with accuracy and precision. It is here that the mystic, scientist, engineer, and business man meet in a melting-pot of chaotic values. In their midst may be heard proposals for taking the cube root of the diameter of the neutrino as the standard for all linear measures, as well as for taking a division of the cubit used in the Great Pyramid, provided it is corrected by a constant of divine origin. Over all may be heard, in rare moments of practical wisdom, the stolid "Nays" of work horses and their owners who are against all proposals with a remarkable lack of favoritism.

The loftiness of the third aim was and still is beyond dispute. Measures must be related to each other in the ratios of the number base. This conception of a great ideal was a marked step forward in man's fight against confusion and inefficiency. Here, at last, was a point upon which exponents of 2, 8, 10, 12, 16, 60 and even 420 base number systems could agree. Death to confusion!

It is with these three axiomatic ideals that the novice in the field of metrological improvement must concern himself. Using these trinary principles to guide one's efforts towards reform and to avoid sin is an a priori must for the St. George with his yardstick lance in his struggle with matter.

It has been suggested that only scientific study of the world in which we live determines our weights and measures, but I favor writing our all embracing ideals on the back of a postage stamp. This may be neatly folded and placed over the navel where one may contemplate it for twelve lunar cycles by monochromatic moonlight. Following this procedure religiously should enable the practitioner to give birth to perfection.



THE FIBONACCI SERIES AND RECIPROCALS

by Harry C. Robert, Jr.

Jerome S. Meyer, in his book "Fun With Mathematics", World Publishing Company, 1952, credits Professor Jekuthiel Ginsburg with pointing out that the reciprocal of 89 (base ten) produces the Fibonacci Series. 8o is the eleventh member of the series written base ten.

Thus:

```
.01
  1
   2
    3
     5
      8
      13
       21
        34
         55
          89
-----------
.0112359550 etc. = 1/89
```

Base Ten

The manner in which this curiosity is presented by Meyer as "the reciprocal of the eleventh Fibonacci number", makes it appear that Mr. Meyer, at least, and possibly Professor Ginsburg, also have missed the rather interesting general case.

The general case has nothing to do with the reciprocal of a Fibonacci number. Thus *75*, the dozen base number corresponding to the base ten 89 does not have the same property. However, *XƐ* does have it -

Thus:

```
.01
  1
   2
    3
     5
      8
      11
       19
        2X
         47
          75
          100
-----------
.0112359303 etc. = 1/XƐ
```

Base Twelve

For any base B, the number $(B^2 - B - 1)$ has this same property. For base eight, base three and base two, $(B^2 - B - 1)$ is a Fibonacci number. For other $B \leqq$ dozen this is not so. The Fibonacci connection appears to be the fact that the equation $(B - B - 1 = 0)$ gives the ratio between successive terms of the Fibonacci Series, that is $\frac{1 - \sqrt{5}}{2}$ and $\frac{1 + \sqrt{5}}{2}$.

If we want our Fibonacci Series spaced two places apart in the reciprocal we use the reciprocal of $(B^4 - B^2 - 1)$.

Thus for the dozen base -

.0001
1
2
3
5
8
11
19

1/ƐXƐƐ = .00010102030508 11 19 etc.

In decimals $\frac{1}{9899}$ = .00010102030508 13 21 etc.

If you prefer three spaces, use $(B^6 - B^3 - 1)$.

If you want only the even terms of the Fibonacci Series, use $(B^2 - 3B + 1)$.

Thus: .01
3
8
19
47
.013X17 etc. 1/91

Using $(B^4 - 3B^2 + 1)$

1/Ɛ901 = .0010308 19 48

and $(B^6 - 3B^3 + 1)$ gives

1/ƐƐ9001 = .0000010030080190471OO . .

All this can be proved by simple algebraic operations. Thus by division -

$$\frac{1}{B^2 - B - 1} = \frac{1}{B^2} + \frac{1}{B^3} + \frac{2}{B^4} + \frac{3}{B^5} + \frac{5}{B^6} + \frac{8}{B^7} + \cdots + \frac{Fn}{B^{n+1}} \quad (n \to \infty)$$

Here we have an interesting algebraic fallacy - a paradox of the infinite -

Let B = 1 in the above expression and we have

$$-1 = 1 + 1 + 2 + 3 + 5 + 8 + \cdots + F_n \quad (n \to \infty)$$

That is, sum of Fibonacci Series = - 1

By letting B = 0 in $(B^2 - B - 1)$ we get another fallacy, $\infty = -1$. By equating the two false values of (-1) we get the obvious true result: Sum = ∞.

I have not determined for what values of B the series may be convergent - but certainly it diverges for all negative values of $(B^2 - B - 1)$. More correctly, it has no sum for these values.

The following has no connection with the Fibonacci numbers, but the same methods can be used to generate other interesting series.

Thus from $(B^2 - 2B + 1)$ 1/X1 = .0123456 ... etc.

from $(B^4 - 2B^2 + 1)$ 1/ƐX01 = .00010203040506O708 etc. ...

From this one can predict that 1/ƐX01 has a cycle of 1ƐX figures

and that 1/ƐƐƐƐX000001 has a cycle of 5ƐƐƐƐ6 figures.

Consecutive powers in the reciprocals are given by the form (Bm - a).

Thus: $$\frac{1}{B^m - a} = \frac{a^0}{B^m} + \frac{a^1}{B^{2m}} + \frac{a^2}{B^{3m}} + \frac{a^3}{B^{4m}} + \cdots + \frac{a^n}{B^{m(n+1)}}$$

Thus: $$\frac{1}{\mathcal{E}9} = .\underline{01}\,\underline{03}\,\underline{09}\;\;\underline{23}\;\;6\;\ldots$$

And for m = 1)

a = 2)

```
   .1
     2
      4
       8
       14
        28
         54
          χ8
          194
           368
   ----------
1/χ = .12497249 etc.
```

Thus the reciprocal of $\mathcal{X}\mathcal{E}$ which we obtained by adding the Fibonacci Series can also be obtained by adding powers of <u>11</u>.

```
   .01
     11
      121
       1331
        14641
         15χχ51
          1749361
   ------------
1/χƐ = .01123593 etc.
```

and the following two series have equal sums -

$$\frac{1}{10^2} + \frac{12}{10^4} + \frac{35}{10^6} + \frac{91}{10^8} + \frac{1\mathcal{E}\mathcal{X}}{10^{\mathcal{X}}} + \cdots + \frac{(10F_{n-1} + F_n)}{10^{n+1}}$$

$$= \frac{1}{10^2} + \frac{11}{10^4} + \frac{11^2}{10^6} + \frac{11^3}{10^8} + \frac{11^4}{10^{\mathcal{X}}} + \cdots + \frac{11^{\frac{n-1}{2}}}{10^{n+1}}$$

PRODUCTS REPEATING INTEGERS OF FACTORS

by George S. Terry

In the Bulletin of Dec. 1947, Vol. 3, No. 4, p. 17, a few decimal examples were given of numbers whose factors were composed of the same numerals as their product. Instances were shown that comparison be made with similar duodecimal cases.

Results depend on congruence modulo Base-less-one (decimally casting out nines). Since integers in the factors are the same as integers in the product, both sum and product of remainders after casting out from factors are the same (mod. B-1) as the remainder after casting out from the product.

$$r_a + r_b \equiv r_a r_b \equiv r_{ab} \mod (B-1)$$

Decimal	Duodecimal
41·35 = 1435	81 · 2Ɛ = 1826
5 + 8 ≡ 5(8) ≡ 4 mod 9	9 + 8 ≡ 9(8) ≡ 6 mod Ɛ

Pairs of remainders thus admissable are:

Decimal (0·0) (2·2) (3·6) (5·8)
Duodecimal (0·0) (2·2) (3·7) (4·5) (6·χ) (8·9)

Two figure decimal factors were studied by Biddle in Math: Questions 1911, p. 60, with the following results:

21·87	21·60	15·93
	41·35	27·81
	51·30	80·86

Corresponding duodecimal results are:

21·χ8	21·70	4χ·Ɛ7
	31·50	60·73
	41·40	62·63
	61·30	
	81·26	
	91·24	

In either list the first column holds for any even Base since the product of (2B-1) with (B-2) B + (B/2 + 2) is $B^3 + (B-2)B^2 + 2B + (B/2 + 2)$.

The second column holds for any composite Base for if $B^2 = pq$ with $p > q$, the product of (p + B) with (Bq + 1) is $B^3 + B^2q + (p + B)$.

The third column holds only for the Base under consideration.

Method for check. Take any congruent pair (a·b) with product just over 1000. Continue adding (B-1) to a and (B-1)a to the product.

For three figure factors, which were not listed by Biddle, we have for a similar column 1, the product of (B-1) (B^2-1) with (B-1 (B^2+1) for any Base., e.g. 891·909 or *XƐ1·Ɛ0Ɛ* .

For column 2, if $B^4 = pg$ with p in two integers, $(Bp+1)$ (B^2+q) gives 12 (*14*) results duodecimally *141·X00* to *X81·216* . Decimally six results from 161·725 to 801·225.

Also if p is a multiple of the Base, $(Bp+1)$ $(B+q)$ gives six results duodecimally from *201·610* to *901·150*. Decimally four results from 201·510 to 801·135.

Other algebraic results may be found from the following decimal and duodecimal lists.

DECIMAL FACTORS

135·801	216·864	281·443	351·900	468·612	630·855
140·926	216·981	281·650	351·909	468·891	641·650
141·840	221·782	284·926	356·431	470·542	650·704
146·938	225·801	296·851	356·926	473·800	650·875
150·930	225·810		360·936	476·941	650·983
152·761	231·534	300·501	366·948	482·890	657·864
152·824	231·543	300·510	369·981	489·582	680·926
156·942	231·588	311·422	371·470	491·845	681·759
158·701	231·750	315·423	381·969		
161·725	231·759	317·425	383·971	524·623	720·936
165·951	240·651	317·461	392·863	530·635	750·906
167·701	246·510	320·926	392·926	533·641	765·963
176·926	248·881	321·678	395·491	533·686	776·992
179·725	248·926	321·975	396·414	534·591	788·926
	251·500	323·410		540·846	
201·510	251·608	323·671	420·678	545·899	800·860
201·600	251·860	323·806	422·581	546·840	807·984
201·627	252·801	327·591	431·707	549·891	825·957
201·897	255·807	336·951	431·725	563·953	843·876
204·516	260·401	338·692	431·878	570·834	855·927
204·615	261·486	341·626	432·891	572·926	858·951
210·501	261·585	342·963	435·870	585·630	875·902
210·600	269·581	350·401	437·602	588·951	891·909
210·870	269·842	350·410	446·908	590·845	891·945
210·906	270·810	350·725	461·524	593·662	894·906
210·915	275·719	351·387	464·926		896·926
215·635	276·822	351·414	465·831	624·780	953·986

DUODECIMAL FACTORS

101·Ɛ91	214·Ɛ03	300·601	38Ɛ·X10	524·994	6X9·Ɛ25
121·Ɛ6X	215·Ɛ09	300·610	391·580	526·607	6XX·Ɛ88
123·Ɛ19	216·X81	300·927	393·410	528·X24	6Ɛ1·X78
134·Ɛ27	227·X01	301·410	393·X6Ɛ	531·8X1	6Ɛ3·955
136·X61	230·951	301·500	395·721	531·Ɛ80	
141·X00	230·Ɛ31	301·60X	397·414	533·7X5	700·9Ɛ5
146·X10	231·640	303·415	3X0·418	541·Ɛ60	709·Ɛ69
150·901	231·948	303·514	3X1·430	544·904	730·935
150·X29	231·Ɛ28	306·X27	3X5·842	545·XX9	753·907
160·X13	236·641	310·401	3X6·X37	546·XX7	759·935
161·900	240·901	310·500		548·X00	765·Ɛ03
161·91X	241·626	310·574	400·401	54Ɛ·973	767·991
167·891	242·8Ɛ1	310·592	400·410	556·Ɛ31	7X4·X61
169·X41	242·Ɛ18	310·826	401·40Ɛ	557·Ɛ0X	7Ɛ7·Ɛ25
170·801	245·XƐ1	315·431	402·802	574·Ɛ31	
177·X42	248·610	319·409	410·627	581·945	800·857
180·X54	248·61Ɛ	31Ɛ·745	416·623	588·ƐX7	812·Ɛ47
182·8X4	250·716	321·460	421·68Ɛ	590·Ɛ61	817·Ɛ40
185·936	252·710	321·46Ɛ	423·973	593·786	828·Ɛ77
185·Ɛ70	253·501	321·550	427·760	595·Ɛ36	850·969
189·941	257·71X	321·55Ɛ	429·6X0	5X0·763	852·961
190·X25	260·801	323·875	430·X96	5X9·Ɛ85	853·X79
191·7X5	260·810	326·Ɛ00	431·XX0		856·X73
194·954	267·781	326·ƐX1	431·Ɛ81	600·631	861·880
198·761	268·Ɛ40	32X·4Ɛ1	432·620	600·730	871·X97
198·ƐX4	271·9Ɛ8	32X·925	432·937	600·73Ɛ	88Ɛ·Ɛ04
1X0·704	276·Ɛ50	330·541	434·93X	600·7Ɛ3	894·X70
1X3·Ɛ99	279·X31	334·Ɛ51	436·XƐ3	602·603	8X7·X80
1X5·933	280·6X1	337·Ɛ02	439·XX6	604·818	
1X9·Ɛ26	280·X34	340·Ɛ21	443·92Ɛ	613·70X	914·Ɛ8X
1XƐ 731	281·515	341·4X6	460·935	620·630	917·X38
1Ɛ3·925	281·560	344·X2X	461·Ɛ00	626·X35	924·Ɛ88
1Ɛ7·X82	281·769	350·513	461·Ɛ0Ɛ	628·924	956·Ɛ71
1Ɛ8·710	281·976	352·510	478·X91	629·Ɛ91	978·987
	284·X80	35X·Ɛ21	479·620	635·Ɛ43	985·X10
200·814	284·Ɛ7Ɛ	360·Ɛ35	495·Ɛ30	636·637	9Ɛ0·X81
200·841	286·Ɛ87	366·Ɛ41	497·Ɛ80	636·691	9Ɛ6·Ɛ41
201·610	288·Ɛ77	369·Ɛ95	499·X01	641·7X5	
201·62X	28X·701	370·Ɛ42	4X0·Ɛ70	641·Ɛ74	X31·ƐX8
201·700	294·691	371·434	4X1·628	645·X60	X65·X99
201·XƐ8	295·Ɛ87	371·Ɛ56	4X2·X50	646·870	X90·Ɛ09
210·601	296·Ɛ0X	378·X31	4X4·X31	646·X50	X91·Ɛ08
210·700	2X1·7Ɛ6	379·929	4X6·ƐX9	649·900	XX5·Ɛ70
210·738	2XX·641	380·461	4Ɛ1·X23	656·XƐƐ	XƐ1·Ɛ0Ɛ
210·X80	2Ɛ0·517	383·421		658·X37	XƐ1·Ɛ56
210·Ɛ07	2Ɛ1·936	384·41Ɛ	500·834	65Ɛ·XƐ1	XƐ9·Ɛ81
210·Ɛ16	2Ɛ2·X6Ɛ	384·Ɛ6X	503·893	663·Ɛ41	
212·X97	2Ɛ3·Ɛ69	385·591	504·Ɛ8Ɛ	67Ɛ·X95	Ɛ38·ƐX1
214·7X8		385·Ɛ40	519·ƐƐ5	6X5·Ɛ42	

CONVERSION MULTIPLES

Numbers, which read duodecimally, are 2, 3, 4 etc., times their value read decimally, are conversion multiples. Four conversion doubles were given in the Bulletin of Dec. 1950 and a complete list of a dozen pairs of such doubles were given by H.K. Humphrey in the Bulletin of Oct. 1951 together with the formula for their discovery.

This is an interesting recreation inasmuch as the result may be directly obtained working either decimally or duodecimally. In fact the office adding machine will give results for higher multiples than doubles with little labor though care is necessary.

Multiples have been found in every case unless, as in the case of trebles, the positive coefficient is greater than ten times the negative ones. See p. 16, Oct. 1952 Bulletin.

Numbers read duodecimally which are N times the same number read decimally.

N = 2. Twenty one results as published in Bulletin.
N = 3. No results possible.
N = 4. *1586 62568 · 1590 06368 · 1645 82768*
N = 5. *11622 75366 · 23748 88768 · 36411 99126*
12096 36884 · 35877 15581 · 36412 13848
12126 13402 · 35906 77288 · 49043 49164
N = 6. Same as for N = 5 with zero added.
N = 7. *13 97886 38673 · 29 67984 28382*
14 55360 09306 · 29 68192 72539
15 09470 51826
N = 8. No results possible.
N = 9. No results possible.
N = *Ten.* *1560 64673 11590*
1615 86397 63478
1615 86406 26638
N = *Twelve.* Same as for N = *Ten* with zero added.
N = *Eleven. Thirteen. Fourteen.* No results possible.
N = *Fifteen.* *2 28616 36780 37713 · 4 67481 70895 11539*
2 33583 38347 10186 · 4 67481 70900 27899
3 48186 15617 37960 · 4 72466 96817 72913
3 52877 57041 35313 · 2 33603 09770 22933
N = *Eighteen.* Same results as for N = *Fifteen* with zero added.

FRACTIONS WITH EQUAL NUMERAL PRODUCTS

by Harry C. Robert, Jr.

(see previous paper on A MATHEMATICAL RECREATION)
(in Bulletin, Vol. 9, No. 1, April 1953, p. 24)

This study concerns certain fractions whose separate products of the numerals in the numerator, - over the products of the numerals in the denominator, - remain of the same value. The earlier report made a brief statement of the problem and gave examples on several bases. The study has been continued, and this paper amends the previous report, detailing the recent development of our explorations.

1. The Base Twelve results stated are found to be incomplete, $\frac{18}{X5}$ and $\frac{26}{X5}$ being omitted.

2. The method given in Scripta is laborious. First step may be to get a simpler method of finding all solutions for various Bases.

3. Let Base = B. Then $\frac{aB + b}{cB \quad d} = \frac{ab}{cd}$

Transposing and regrouping $B\left(\frac{1}{b} - \frac{1}{d}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)$ and since

$b > d$ and $c > a$, both sides are negative so we change signs:-

$$B\left(\frac{1}{d} - \frac{1}{b}\right) = \left(\frac{1}{a} - \frac{1}{c}\right)$$

A small table of differences between reciprocals is all that we need to obtain solutions for any Base. The Scripta method has to be repeated for each Base. The small table on the next page suffices to Base Sixteen.

4. The method of using the table is simple and can be used for any Base. Let us take $\frac{1}{18}$ and test for results on various Bases.

Table of Values of $\left(\frac{1}{d} - \frac{1}{b}\right)$ and $\left(\frac{1}{a} - \frac{1}{c}\right)$

Values of 1/b and 1/c	Values of 1/a and 1/d: 1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/X	1/Ɛ	1/10	1/11	1/12
1/2	1/2													
1/3	2/3	1/6												
1/4	3/4	1/4	1/10											
1/5	4/5	3/X	2/13	1/18										
1/6	5/6	1/3	1/6	1/10	1/26									
1/7	6/7	5/12	4/19	3/24	2/2Ɛ	1/36								
1/8	7/8	3/8	5/20	1/8	3/34	1/20	1/48							
1/9	8/9	7/16	2/9	5/30	4/39	1/16	2/53	1/60						
1/X	9/X	2/5	7/26	3/18	1/X	1/13	3/5X	1/34	1/76					
1/Ɛ	X/Ɛ	9/1X	8/29	7/38	6/47	5/56	4/65	3/74	2/83	1/92				
1/10	Ɛ/10	5/10	1/4	1/6	7/50	1/10	5/70	1/20	1/30	1/50	1/Ɛ0			
1/11	10/11	Ɛ/22	X/33	9/44	8/55	7/66	6/77	5/88	4/99	3/XX	2/ƐƐ	1/110		
1/12	11/12	3/7	Ɛ/36	5/24	9/5X	2/19	1/12	3/48	5/X6	1/2Ɛ	3/10X	1/70	1/132	
1/13	12/13	11/26	4/13	Ɛ/50	2/13	1/X	8/89	7/XX	2/39	1/26	4/119	1/50	2/143	1/156

(All figures - Base Twelve)

$\frac{1}{18}$ is in $\begin{cases}\text{line } 1/5\\ \text{column } 1/4\end{cases}$ so a solution may be $\frac{a5}{c4}$ on any Base.

Multiply by the Base (expressed duodecimally since the table is so expressed) we have:-

Base Six, $\frac{6}{18} = \frac{3}{X}$ which occurs in $\begin{cases}\text{column } 1/2\\ \text{line } 1/5\end{cases}$ Solution $\frac{25}{54}$

Base Ten, $\frac{X}{18} = 1/2$ which occurs in $\begin{cases}\text{column } 1/1\\ \text{line } 1/2\end{cases}$ Solution $\frac{15}{24}$

Base Twelve $\frac{10}{18} = 3/5$ which does not occur.

Base Fifteen $\frac{13}{18} = 3/4$ which occurs in $\begin{cases}\text{column } 1/1\\ \text{line } 1/4\end{cases}$ Solution $\frac{15}{44}$

Base Sixteen $\frac{14}{18} = 4/5$ which occurs in $\begin{cases}\text{column } 1/4\\ \text{line } 1/5\end{cases}$ Solution $\frac{45}{54}$

For any Base we use only the lines above the reciprocal $1/_{Base}$

and the columns to the left of the one headed $1/_{B-1}$.

We only have to multiply those fractions which are smaller than $1/_B$ for each Base, since all values in the table of solutions are less than unity. Reciprocals of these solutions follow as a matter of course.

The small number of fractional differences to be checked for each Base are near the right side of the table. After multiplying by B, the corresponding fraction will be found near the left of the table. All computations can be made Base Twelve, converting individual integers, if necessary, as the last operation. Base Ten could be used for the table in same way if desired.

5. The formula in the Bulletin, though incomplete, may be the most general formula that can be written for this interesting bit of "trivia".

It might be preferable to write this general formula as follows:

$$\frac{\left[\frac{B}{M}-1\right]\left[B-N\right]}{\left[\frac{B}{N}-1\right]\left[B-M\right]} = \frac{N}{M}$$

where M and N are all of the divisors of B such that

$$B > M > N \geq 1$$

If there are "n" such divisors, including 1, there will be a total of $\frac{n\,(n-1)}{2}$ solutions given by this formula.

6. It appears that when B is prime there are no solutions given either by the general formula or otherwise. Also note that no prime, Q, $\frac{B+1}{2} > Q > B - 1$, appears as an integer in any of the listed solutions.

7. When B is odd, all four integers in solutions given by the general formula are even. Dividing each by 2 gives another solution. In other cases where the four integers are not relatively prime, other divisors can also be used.

8. When B = 6p - 2, there are three solutions of the form.

$$\frac{(p)\ (3p)}{(3p)\ (3p-1)} \qquad \frac{(1)\ (3p)}{(3p)\ (2p)} \qquad \frac{(1)\ (3p-1)}{(p)\ (2p)}$$

Note: Composite values of p might be expected to extend number of solutions, but this does not appear to be the case. Should be checked on large Bases.

When p = 1 B = 4, the above gives only one solution. For all other cases, all three solutions appear.

For Base Ten, p = 2, and we get three of the Base Ten solutions not given by the General Formula. That is,

$$\frac{26}{65}$$

$$\frac{16}{64} \qquad \frac{15}{24}$$

9. When B = 10p + 4, there are three solutions of the form.

$$\frac{(p)\ (4p+1)}{(4p+1)\ (4p)}$$

$$\frac{(1)\ (4p+1)}{(4p+1)\ (3p+1)} \qquad \frac{(1)\ (4p)}{(p)\ (3p+1)}$$

Note diagonal arrangement same as for General Form.

When p = 1, there is one solution. $p \geq 2$ gives all three. These solutions can be multiplied by 2 to give another set of solutions. When multiplied by 3, solutions are identical with some given by the General Formula.

X. Solutions for Various Bases.

Base 4 (1 solution)	$\frac{13}{32}$	General Formula and also by the special forms for B = (6n - 2)
Base 6 (3 solutions)	$\frac{25}{54}$; $\frac{15}{53}$ $\frac{14}{23}$	Given by General Formula.
Base 8 (4 solutions)	$\frac{37}{76}$; $\frac{17}{74}$ $\frac{16}{34}$	Given by General Formula.
	$\frac{14}{33}$	Apparent special case
Base 9 (4 solutions)	$\frac{28}{86}$	General Formula
	$\frac{14}{43}$	Above divided by 2
	$\frac{16}{44}$ $\frac{26}{55}$	Apparent special cases

Base $\mathcal{X}$ (7 solutions)

General Formula: $\frac{49}{98}$; $\frac{19}{95}$ $\frac{18}{45}$

Special Form for B = $6p - 2$: $\frac{26}{65}$; $\frac{16}{64}$ $\frac{15}{24}$

Apparent special case: $\frac{14}{63}$

Base 10 (11 solutions)

General Formula. Note diagonal arrangements:
$\frac{5\mathcal{E}}{\mathcal{E}\mathcal{X}}$;
$\frac{3\mathcal{E}}{\mathcal{E}9}$ $\frac{3\mathcal{X}}{59}$;
$\frac{2\mathcal{E}}{\mathcal{E}8}$ $\frac{2\mathcal{X}}{58}$ $\frac{29}{38}$;
$\frac{1\mathcal{E}}{\mathcal{E}6}$ $\frac{1\mathcal{X}}{56}$ $\frac{19}{36}$ $\frac{18}{26}$

Appear to be Special Cases of two different types.: $\frac{39}{68}$; $\frac{18}{\mathcal{X}5}$ $\frac{26}{\mathcal{X}5}$

Base 12 (8 solutions)

General Formula: $\frac{6\ (11)}{(11)\ (10)}$; $\frac{1\ (11)}{(11)\ 7}$ $\frac{1\ (10)}{6\ 7}$

Appear to be special cases of two types.: $\frac{28}{47}$; $\frac{17}{55}$ $\frac{27}{66}$; $\frac{2\ (10)}{9\ 9}$ $\frac{3\ (10)}{\mathcal{X}\ \mathcal{X}}$

Base 13 ($\mathcal{E}$ solutions)

General Formula: $\frac{4\ (12)}{(12)\ (10)}$; $\frac{2\ (12)}{(12)\ \mathcal{X}}$ $\frac{2\ (10)}{4\ \mathcal{X}}$

General Form divided by 2: $\frac{27}{76}$; $\frac{17}{75}$ $\frac{16}{25}$

Apparent special cases - at least two types: $\frac{19}{66}$; $\frac{2\mathcal{X}}{39}$ $\frac{2\mathcal{X}}{88}$ $\frac{39}{88}$ and $\frac{15}{44}$

Base 14 (10 solutions)

General Form: $\frac{7\ (13)}{(13)\ (12)}$; $\frac{3\ (13)}{(13)\ (10)}$ $\frac{3\ (12)}{7\ (10)}$; $\frac{1\ (13)}{(13)\ 8}$ $\frac{1\ (12)}{7\ 8}$ $\frac{1\ (10)}{3\ 8}$

Special Form for B = $6p - 2$: $\frac{39}{98}$; $\frac{19}{96}$ $\frac{18}{36}$

* $\frac{15}{54}$ **Special Form for B = $10p + 4$**

$\frac{2\mathcal{X}}{\mathcal{X}8}$ **2-Multiple of above**

$\frac{3\ (10)}{(13)\ \mathcal{X}}$ **Apparent special case**

* Can also be obtained from General Form $\frac{3\ (13)}{(13)\ (10)}$ by dividing same by $3/3$.

ℰ. Note that the arrangement of General and Special Form solutions all show a consistent pattern on the diagonals of the four integers.

10. The Special Cases in nearly every case have three integers transposed or rotated from one of the regular forms. In nearly every case the denominators of "Specials" contain high powers. Every denominator of "Specials" contains at least a square.

11. Two particular patterns of specials are noted, but not all specials fall in these categories.

(a) One case exists where a previous solution which may be either regular or special itself, furnishes two numerators for the same denominator to produce two additional solutions. Thus -

Base *10* $\frac{18}{26}$ (regular) gives $\frac{18}{χ5}$ and $\frac{26}{χ5}$

Base *13* $\frac{2χ}{39}$ (special) gives $\frac{2χ}{88}$ and $\frac{39}{88}$

(b) The second definite pattern results in adding *10* to the numerator and *11* to the denominator to produce a second special solution. The denominator integers are equal for this pattern. Thus -

Base *9* $\frac{16}{44}$ gives $\frac{26}{55}$

Base *12* $\frac{17}{55}$ gives $\frac{27}{66}$

and $\frac{2\ (10)}{9\ \ 9}$ $\frac{3\ (10)}{χ\ \ χ}$

Base *13* $\frac{19}{66}$ by adding $\frac{20}{22}$ gives $\frac{39}{88}$ which is one of the other special patterns.

COMPOSITE SQUARE