COUNTING IN DOZENS

1	2	3	4	5	6	7	8	9	X	Ɛ	10
one	two	three	four	five	six	seven	eight	nine	dek	el	do

Our common number system is decimal - based on ten. The dozen system uses twelve as the base, which is written *10*, and is called *do*, for dozen. The quantity *one gross* is written *100*, and is called *gro*. *1000* is called *mo*, representing the meg-gross, or great-gross.

In our customary counting, the places in our numbers represent successive powers of ten; that is, in 365, the 5 applies to units, the 6 applies to tens, and the 3 applies to tens-of-tens, or hundreds. Place value is even more important in dozenal counting. For example, *265* represents *5* units, *6* dozen, and *2* dozen-dozen, or gross. This number would be called *2 gro 6 do 5*, and by a coincidence, represents the same quantity normally expressed as 365.

Place value is the whole key to dozenal arithmetic. Observe the following additions, remembering that we add up to a dozen before carrying one.

94	136	Five ft. nine in.	5.9'
31	694	Three ft. two in.	3.2'
96	3Ɛ2	Two ft. eight in.	2.8'
19Ɛ	1000	Eleven ft. seven in.	Ɛ.7'

You will not have to learn the dozenal multiplication tables since you already know the 12-times table. Mentally convert the quantities into dozens, and set them down. For example, 7 times 9 is 63, *which is* 5 dozen and 3; so set down *53*. Using this "*which is*" step, you will be able to multiply and divide dozenal numbers without referring to the dozenal multiplication table.

Conversion of small quantities is obvious. By simple inspection, if you are 35 years old, dozenally you are only *2Ɛ*, which is two dozen and eleven. For larger numbers, keep dividing by 12, and the successive remainders are the desired dozenal numbers.

```
12 ) 365
12 ) 30  + 5
12 ) 2   + 6
     0   + 2      Answer: 265
```

Dozenal numbers may be converted to decimal numbers by setting down the units figure, adding to it 12 times the second figure, plus 12^2 (or 144) times the third figure, plus 12^3 (or 1728) times the fourth figure, and so on as far as needed. Or, to use a method corresponding to the illustration, keep dividing by *X*, and the successive remainders are the desired decimal number.

Fractions may be similarly converted by using successive multiplications, instead of divisions, by 12 or *X*.

Numerical Progression

1	One		
10	Do	.1	Edo
100	Gro	.01	Egro
1,000	Mo	.001	Emo
10,000	Do-mo	.000,1	Edo-mo
100,000	Gro-mo	.000,01	Egro-mo
1,000,000	Bi-mo	.000,001	Ebi-mo
1,000,000,000	Tri-mo	and so on.	

Multiplication Table

1	2	3	4	5	6	7	8	9	X	Ɛ
2	4	6	8	X	10	12	14	16	18	1X
3	6	9	10	13	16	19	20	23	26	29
4	8	10	14	18	20	24	28	30	34	38
5	X	13	18	21	26	2Ɛ	34	39	42	47
6	10	16	20	26	30	36	40	46	50	56
7	12	19	24	2Ɛ	36	41	48	53	5X	65
8	14	20	28	34	40	48	54	60	68	74
9	16	23	30	39	46	53	60	69	76	83
X	18	26	34	42	50	5X	68	76	84	92
Ɛ	1X	29	38	47	56	65	74	83	92	X1

The Duodecimal Bulletin

Volume 5 - No. 2
September, 1949

THE DUODECIMAL SOCIETY OF AMERICA

20 Carlton Place ~ ~ ~ ~ ~ Staten Island 4, N. Y.

THE DUODECIMAL SOCIETY OF AMERICA

is a voluntary nonprofit organization for the conduct of research and education of the public in the use of Base Twelve in numeration, mathematics, weights and measures, and other branches of pure and applied science.

Full membership with voting privileges requires the passing of elementary tests in the performance of twelve-base arithmetic. The lessons and examinations are free to those whose entrance applications are accepted. Remittance of $6, covering initiation fee ($3) and one year's dues ($3), must accompany applications.

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WHY CHANGE?

This same question was probably rife in Europe between the years 1000 and 1500, when the new Arabic numerals were slowly making their inching progress in displacing the comfortable and familiar Roman numbers universally used. *"Why even try to learn to use these heathenish scrawly symbols, with their stubborn propensity for error, in place of the beautiful clear numbers which our fathers have used for untold generations? Think of the needless waste. We would have to change all of our counting boards and abaci. X is X, isn't it? And why do we need a symbol for nothing? You can't count it! No! Let us keep to our simple tried and true numerals, and let the barbarians scratch their heads, and rub themselves out. It will all come to 0 anyhow."*

Yet, although it took D years, the new notation became generally used, and man's thinking leapt forward like an arrow sped from a bow. The early years of the Renaissance marked a new stage in the use of symbols, with the advent of algebra, fractionals (decimals?), logarithms, analytical geometry, and the calculus. Can you imagine what it would be like to try to express the coordinates of points on a curve in Roman notation?

CONTENTS

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Mathematicians became conscious of a new dimension in symbolism, and the fundamental concepts of number were re-examined. Man awoke to the fact that different number bases could be used, and Simon Stevin stated in 1585 that the duodecimal base was to be preferred to the decimal.

The new Arabic notation accomodated mathematical statement better, and facilitated ideation. All thinking accelerated when released from the drag of the cumbrous Roman notation.

The parallel seems tenable. The notation of the dozen base accomodates mathematical statement better, and facilitates ideation. It, too, is a step forward in numerical symbolism. The factorable base embodies a concurrent analysis and definition of numbers that stimulates classification and generalization. Yet this is accomplished by such simple means that students in the primary grades easily learn to perform computations in duodecimals, and can tell why they are better. Literally, the decimal base is unsatis-factory because it has "not-enough-factors."

Then, shouldn't we change? No! No change should be made, and we urge no change. All the world uses decimals. But people of understanding should learn to use duodecimals to facilitate their thinking, and to ease the valuative processes of their minds. Duodecimals should be man's second mathematical language. They should be taught in all the schools. In any operation, that base should be used which is most advantageous, and best suited to the work involved. We expect that duodecimals will progressively earn their way into general popularity. But no change should be made. Perhaps by the year 2000, or maybe by *1200*, which is *14* years later, duodecimals may be the more popular base. But then no change need be made, because people will already be using the better base.

When one is familiar with duodecimals, a number of accessory advantages become apparent. Percentage is a very useful tool, but many percentages come out in awkward figures because of the inflexibility of decimals. When based on the gross, twice as many ratios come out in even figures, and among them are some of those most used, as thirds, sixths, and twelfths, - eighths and sixteenths. There are advantages associated with time and the calendar. Monthly interest rates or charges are derived from annual rates, or the reverse, by simply moving the unit (decimal?) point. The price of a single item bears the same relation to the price of the dozen, and so does the inch to the foot.

The proper correlation of weights and measures has always been one of the world's serious problems. None of the present systems is completely satisfactory. The American and English standards are convenient to use since they are the final result of a long process of practical evolution in which many inconvenient measures have been adjusted or abandoned. The French decimal metric measures have the advantage of being set upon the same base as the number system, and are well systemized. But many of the units are awkward because of their arbitrary sizes, and because their decimal scale does not accomodate division into thirds and fourths readily.

The duodecimal system of weights and measures, based on the inch and yard, the pint and the pound, has the desirable elements of both systems, and few of their faults. This Do-Metric System retains the familiar units of the American and British standards in approximately their present size, and arranges them into an ordered metric system using the scale of twelve. This fits perfectly into the duodecimal notation, and the combination accomodates the inclusion of the units of time and of angular measure within the system, which hitherto has not been possible.

If "playing with numbers" has sometimes fascinated you, if the idea of experimenting with a new number base seems intriguing, if you think you might like to be one of the adventurers along new trails in a science which some have thought staid and established, and without new trails, then whether you are a professor of mathematics of international reputation, or merely an interested pedestrian who can add and subtract, multiply and divide, your membership in the Society may prove mutually profitable, and is cordially invited.

The Duodecimal Bulletin

All figures in italics are duodecimal.

AN EXPONENTIAL EXPRESSION FOR MUSIC

by Trenchard More, Jr.

INTRODUCTION Let me first congratulate Velizar Godjevatz on his new musical notation. After taking ten years of piano lessons, I am still pitifully slow at reading, and perhaps now I will never bother to learn that impossible collection of old conflicting notations. The new stave should bring the pleasure of rapid musical reading within the scope of those who are unable to practice four to eight hours a day. I have tried writing music solely in dozenal numbers, but this presents too much detail for the eye to grasp quickly. Because of its greater simplicity, the Notation Godjevatz lies closer to beauty than the old musical script.

DIFFICULTIES However, the dozenal numbers representing Godjevatz's "Absolute Pitch" remain mathematically arbitrary, as well as the "Audition Range", most existing pitch standards, and most existing notations. Although the idea of numbering notes is a good one, the present duodecimal musical numbers are a logical sequence of symbols rather than a powerful mathematical tool. The lettering of notes appears to be based on habit rather than necessity, both in the new and old notations, and when these letters evolve into such forms as f_3♯♯♯ or D,±A,±F,♯ or e'''♭♭, the spirit of simplicity revolts. In the field of theoretical music there are apparently as many different alphabetic notations as there are experimenters.

OBJECTIVE If possible, we must adopt in our duodecimal simplification of science a system of music satisfactory to the artist, physicist, and mathematician. This system should be able to contain and encourage future developments in music without having to be discarded for a different method. It should be thoroughly grounded, unified, and flexible.

DEFINITIONS Sound is divided into the two categories of noise and musical tone, which are distinguished by periodicity. Noise has no period; that is, it does not repeat itself in constant intervals of time. A musical tone has Force (the amplitude of the sonorous body), Pitch (the rate of periodicity), and quality (how the sonorous body vibrates within each period). We are concerned only with absolute pitch (pitch standards and pitches of musical instruments) and relative pitch (musical intervals octave, fourth, fifth, etc.). We define a cycle as a period, and frequency as the rate of periodicity or the number of cycles per unit time. A musical interval is a ratio of the frequencies of two notes in cycles per unit time. A chord is composed of two or more intervals, or three or more musical tones.

BASIC IDEA It is necessary to select a suitable musical interval as the standard for all other intervals. The ear has the peculiar property that it enjoys simple numerical ratios; such as 1:2, 2:3, 3:5, etc. Of these, the simplest, and most used interval in music is the frequency ratio of 1:2, or the octave. The choice of the octave as the standard interval is the only arbitrary assumption in this system. From it, the notation and all else is derived mathematically. Of course there are many discriminations to be made, but these will be based on our first assumption. The choice of a unit of time will be mentioned later.

The curve of pitch must double itself in every progressive octave. Therefore,

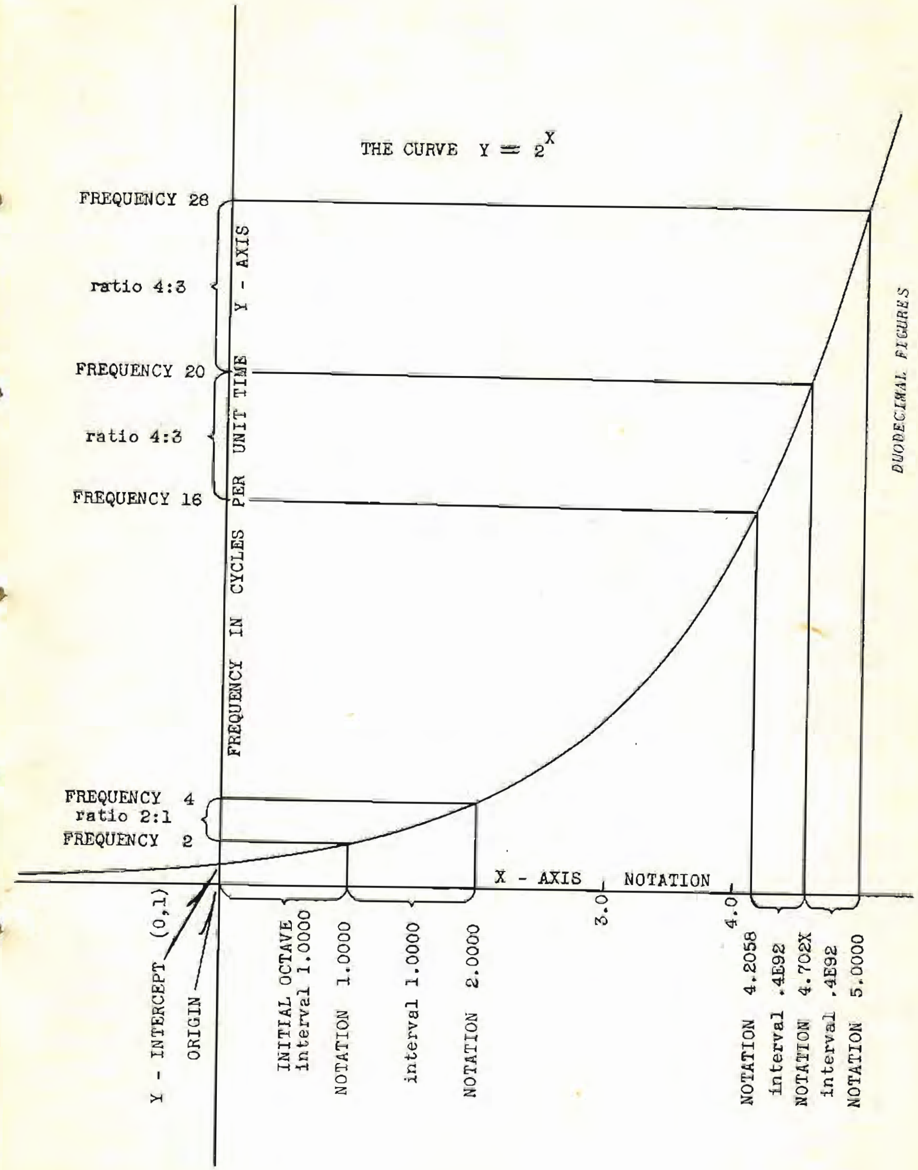
$$y = 2^x \qquad (1)$$

where y is the frequency in cycles per unit time (pitch), and x is the number of octaves (intervals of 1:2). Equation (1) represents an exponential curve which intercepts the y (ordinate) axis at the point (x = 0, y = 1). The curve lies above the x axis, and approaches it asymptotically to the left. We need only that portion of the curve which lies in the first quadrant, upper right. At present, an arbitrary point on this curve is used for a standard, which has the decimal coordinates (8.781, 440). This standard leads to a complicated formula for the calculation of the American Equally Tempered Chromatic Scale, to which most pianos are tuned.

$$y = 2^{\log_2 440 - \frac{N}{12}} \qquad (2) \text{ dec}$$

where y equals the frequency in cycles per second, and N equals the number of semitones above or below A, 440. For the semitones below 440, N becomes negative.

It seems better to avoid arbitrary constants and to retain the simple form of equation (1) by letting our standard point on the curve be the y intercept (0, 1); (grounded). Then all frequencies, absolute and relative, may be represented directly by exponents of 2; (unified). Instead of multiplying and dividing frequencies, we add and subtract their exponents. We will abolish both the alphabetic notation for absolute pitch, and the Roman numeral notation for relative pitch by using their exponents only. Then any absolute pitch, and any musical interval may be represented to any desired degree of accuracy by an exponent of 2; (flexible). Here there are no constants, standards, and symbolic notations to hamper accuracy and expression. There are no arbitrary and complicated conversion methods to



apply between absolute and relative pitch, notation and actual frequency, or one notation and another. The exponential expression for music is built as closely as possible about the curve $y = 2^x$. If those who advance further in the field do the same, unification may be preserved. I hope that this curve is basic and flexible enough to accommodate the now unforeseen advances in music.

INTERVALS Since scales are based on intervals, the interval is mentioned first. We wish to find an exponential representation for the frequency ratio of v : w cycles per unit time. This is given by the equation

$$\frac{v}{w} = 2^{\frac{\log v - \log w}{\log 2}} \qquad v > w > o \qquad (3)$$

where the log may have any convenient base, the most suitable for this paper being do (one dozen). Then all intervals of the form v : w will be represented by the numerical equivalent of $\frac{\log v - \log w}{\log 2}$ instead of arbitrary names, numerals, or letters. The table "Names and Ratios of Intervals" has been calculated for the more prominent musical intervals. Try adding the pure fourth and fifth together, or the major third and the minor sixth, or the minor third and the major sixth, or the major second and minor subdominant seventh, and you will note the ease with which these intervals may be handled.

The "Interval Chart" may be of aid to the experimenter. To find the interval for the ratio of *3 : 8* or *8 : 3*, trace the diagonal leading from *3* to the point where it intersects the diagonal leading from *8*. Immediately to the right of the arrowhead formed by these two diagonals is the answer *1.4Ɛ92*. Every other diagonal has been omitted to ease the eye, so that every whole number on the left has two diagonals leading from it; one blank; and one drawn. Of course this table may be extended, but nearly all of the useful intervals are included within the combinations of the first dofour *(14)* digits. The experimenter may find some very pretty "runs" in this triangle by playing each interval of a diagonal in succession. There are also several arithmetical properties in the table which increase its intrigue, and facilitate its computation.

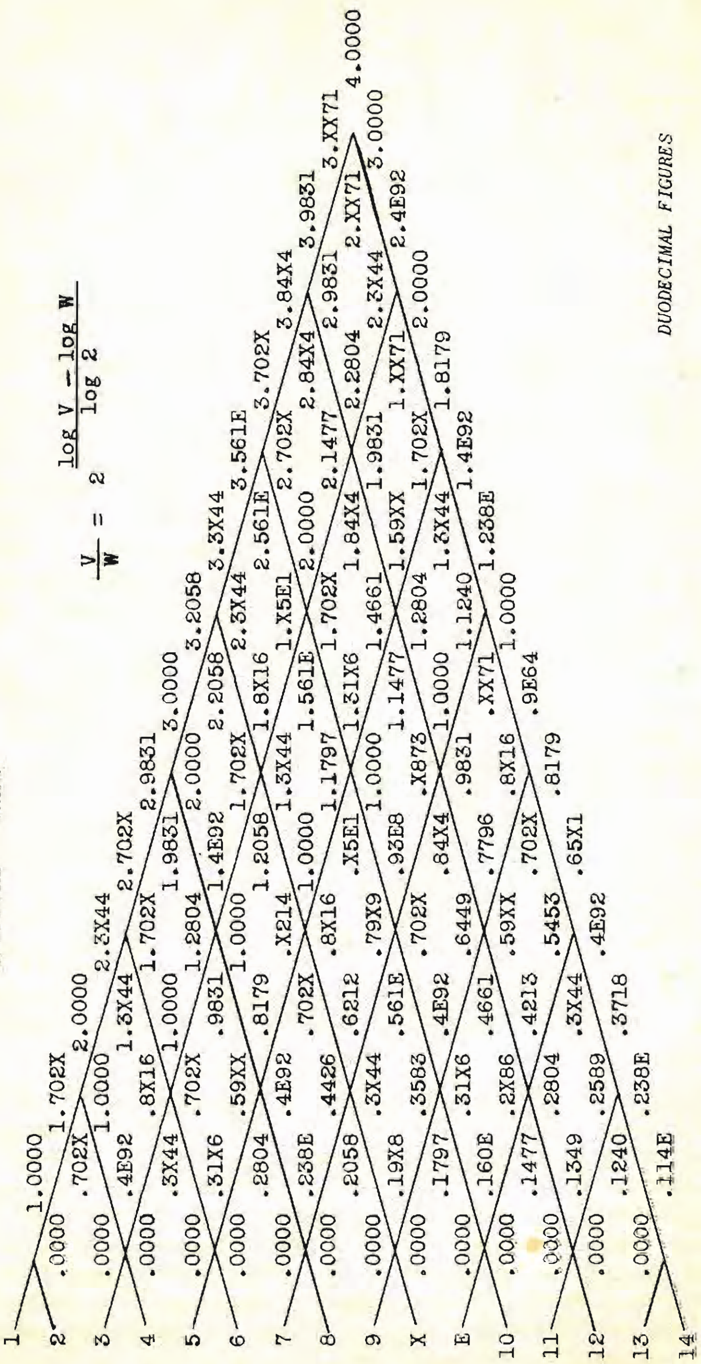
SCALES A scale is a series of absolute pitches which double their frequencies in each progressive octave. Thus each octave of notes is a duplicate of the next, except at a different level of pitch. The notation for absolute pitch is given directly by equation (1), and is more easily calculated by equation (4).

NAMES AND RATIOS OF INTERVALS

Interval	*Ratio*	*Name and Description*
.0000	1 : 1	unison prime partial tone or harmonic
.0015344		cent 1.000 5755 : 1 used in physics; 1200th root of 2
.1000		equal (tempered) semitone 1.059 461 : 1; dozenth root of 2
.114Ɛ	14:13	just (true) semitone
.19X8	X:9	just minor tone
.2000		equal second sixth root of 2
.2058	9 : 8	just major second
.238Ɛ	8 : 7	supersecond
.2804	7 : 6	subminor third
.3000		equal minor third fourth root of 2
.31X6	6 : 5	just minor third
.3X44-	5 : 4	just major third
.4000		equal major third cube root of 2
.4426	9 : 7	supermajor third
.4Ɛ92	4 : 3	just and Pythagorean fourth
.5000		equal fourth dozenth root of 2^5
.59XX	7 : 5	subminor fifth
.6000		equal tritone or augmented fourth square root of 2
.7000		equal fifth dozenth root of 2^7
.702Ɛ	3 : 2	just and Pythagorean fifth
.8000		equal minor sixth cube root of 2^2
.8179	8 : 5	just minor sixth
.8Ɛ16	5 : 3	just major sixth
.9000		equal major sixth fourth root of 2^3
.93X8	10: 7	supermajor sixth
.9831	7 : 4	subminor seventh
.9Ɛ64	14: 9	minor seventh used in the subdominant
.X000		equal minor seventh sixth root of 2^5
.X214	9 : 5	acute minor seventh
.XX71	13:8	just major seventh
.Ɛ000		equal major seventh dozenth root of $2^Ɛ$
1.0000	2 : 1	octave second partial tone
1.2058	9 : 4	ninth
1.2804	7 : 3	subminor tenth
1.31X6	10: 5	minor tenth
1.3X44-	5 : 2	major tenth
1.4Ɛ92	8 : 3	eleventh
1.702X	3 : 1	twelfth third partial tone
1.8179	14: 5	minor thirteenth
1.8X16	X : 3	major thirteenth
1.9831	7 : 2	subminor fourteenth
2.0000	4 : 1	double octave fourth partial tone
2.3X44-	5 : 1	fifth partial tone
2.702X	6 : 1	sixth partial tone
		etc.

INTERVAL CHART

$$\frac{V}{W} = 2^{\frac{\log V - \log W}{\log 2}}$$



DUODECIMAL FIGURES

$$y = 2^x \qquad (1)$$

$$x = \log_2 y = \frac{\log y}{\log 2} \qquad (4)$$

To find the notation for *194.00* cycles per unit time, substitute *194* for y in equation (4), and receive the answer x = *8.0*. Then 2^8 = *194*, and in this exponential expression for music the absolute pitch of *194.00* cycles per unit time will always be written as *8.0* instead of $C_4\flat\flat$ or *194* or $B_3\sharp$. The octaves above this note would be written as *9.0*, *𝒳.0*, etc., and the frequencies of these notes would be *368.00*, *714.00*, etc. A fifth above *𝒳.0* would be written as *𝒳.702𝒳*, and a tempered fifth above *𝒳.0* would be *𝒳.7000*. Thus by adding intervals to pitches, we get new pitches; all in a unified notation.

To return to intervals for a moment, suppose we have the ratio *4:3*, this is the same as *1.4:1*. Calculating our notation for *1.4* cycles... *0.4Ɛ92*, and subtracting our notation for 1 cycle... *0.0000*, we have *.4Ɛ92*, which expresses both the interval of *4:3*, and the absolute pitch of *1.4* cycles. Then intervals and pitches may be thought of as the same, except that the useful intervals will range from *0.0* to *4.0*, and the useful pitches will range from *4.0* upward; that is, if we use a unit of time equal to the second. More about time later.

The "just" or "scientific" major scale is composed of the following ratios: *9:8*, *5:4*, *4:3*, *3:2*, *5:3*, *13:8*, and *2:1*, which complete one octave. The intervals of these ratios determined from the Interval Chart establish the initial octave ranging from *0.0* to *1.0*. To find the notation for the eighth octave, add *8.0* to each interval of the initial octave. Then if it is necessary to find the frequency directly from the notation, multiply the notation (exponent) by log 2, and take the antilogarithm of the product (see equa. 4). With the use of the log log scales on the proposed duodecimal slide rule, a large portion of the curve y = 2^x could be laid out in infinite graduation with but one setting of the slide, thus making the calculations between frequency and notation a matter of tabulation.

TEMPERAMENT Scales are the result of compromises between manipulation and harmony. If we were to play in only one key, matters would be simple, but instead we prefer to play in two tonal modes, and in as many keys as there are notes in the resulting octave; all of this in accurate just intonation. Each key would require new notes, which would require new keys, etc., perhaps making the keyboard octave several feet wider than a grand piano. Obviously temperament is necessary; a subject which has been discussed since the time of the Greeks, who had seven tonal modes instead of two.

The interval of next importance to the octave is the fifth. If we add a dozen fifths together, we have an interval of *7.0299*, which differs from an interval of 7 octaves by an error of *.0299*. Then distributing this error evenly among the dozen fifths, we have the tempered fifth of *.7000*, differing from the true fifth by the error *.00299*. This then is an argument for our modern tempered scale, for an interval of *.002X* (two decimal cents) is scarcely perceptible to the most trained ear. Equal temperament may be thought of as the reduction of every interval to the nearest dozenth place. The error column in "Just and Tempered Scales" shows that just thirds and sixths suffer most from temperament. Now add 1.0, 2.0, 3.0, etc., to each interval in the tempered octave, and you have the notation for the Duodecimal Equally Tempered Chromatic Scale, a notation similar to the one which Godjevatz suggested.

JUST AND TEMPERED SCALES

just	major	just	minor	others		tempered	error
1:1	.0000	1:1	.0000			.0000	.0000
				14:13	.114Ɛ	.1000	-.014Ɛ
9:8	.2058	9:8	.2058			.2000	-.0058
		6:5	.31X6			.3000	-.01X6
5:4	.3X44-					.4000	+.0178+
4:3	.4Ɛ92	4:3	.4Ɛ92			.5000	+.002Ɛ
				7:5	.59XX	.6000	C.0212
				X:7	.6212		
3:2	.702X	3:2	.702X			.7000	-.002X
		8:5	.8179			.8000	-.0179-
5:3	.8X16					.9000	+.01X6
		9:5	.X214	14:9	.9Ɛ64	.X000	+.0058
13:8	.XX71					.Ɛ000	+.014Ɛ
2:1	1.0000	2:1	1.0000			1.0000	.0000

TIME Until now, the customary phrase 'cycles per second' has been carefully substituted by 'cycles per unit time'. This was done to emphasize that only one arbitrary assumption was made (the choice of the octave, all else derived mathematically), and that the curve $y = 2^x$ is independent of time. You may seat yourself at the piano now, without regard to absolute pitch or time, and play the diagonals of the "Interval Chart" by taking each interval to the nearest dozenth place. You may use the tables "Names and Ratios of Intervals", "Just and Tempered Scales",

DUODECIMAL EQUALLY TEMPERED CHROMATIC SCALE $y = 2^x$

graciously checked by Mr. George S. Terry

x	y	x	y	x	y
0.0	1.0000	4.0	14.000	8.0	194.00
0.1	1.0869	4.1	14.Ɛ50	8.1	1X7.28
0.2	1.1577	4.2	15.Ɛ62	8.2	1ƐƐ.42
0.3	1.232Ɛ	4.3	17.03Ɛ	8.3	214.53
0.4	1.3152	4.4	18.1XX	8.4	22X.66-
0.5	1.4027	4.5	19.436-	8.5	245.88
0.6	1.4Ɛ79	4.6	1X.764	8.6	262.06-
0.7	1.5Ɛ91	4.7	1Ɛ.Ɛ81	8.7	27Ɛ.69
0.8	1.7070	4.8	21.494	8.8	29X.46-
0.9	1.8222	4.9	22.XXX	8.9	2ƐX.66-
0.X	1.946Ɛ	4.X	24.613	8.X	320.18
0.Ɛ	1.X7X0	4.Ɛ	26.254	8.Ɛ	343.32
1.0	2.0000	5.0	28.000	9.0	368.00
1.1	2.1516	5.1	29.XX0	9.1	392.54
1.2	2.2Ɛ32	5.2	2Ɛ.Ɛ04	9.2	3ƐX.85
1.3	2.465Ɛ	5.3	32.07X	9.3	428.X6-
1.4	2.62X3	5.4	34.399	9.4	459.0Ɛ
1.5	2.8052	5.5	36.86Ɛ	9.5	48Ɛ.53
1.6	2.9Ɛ36	5.6	39.308	9.6	504.0Ɛ
1.7	2.ƐƐ62	5.7	3Ɛ.Ɛ42	9.7	53Ɛ.17
1.8	3.2121	5.8	42.969	9.8	578.90
1.9	3.4443	5.9	45.998	9.9	5Ɛ9.0Ɛ
1.X	3.691Ɛ	5.X	49.026	9.X	640.34
1.Ɛ	3.9381	5.Ɛ	50.4X9	9.Ɛ	686.64
2.0	4.0000	6.0	54.000	X.0	714.00
2.1	4.2X30	6.1	57.980	X.1	764.X8
2.2	4.5X66	6.2	5Ɛ.X07	X.2	7Ɛ9.4X
2.3	4.90ƐX	6.3	64.139	X.3	855.90
2.4	5.0587	6.4	68.775	X.4	8Ɛ6.1Ɛ
2.5	5.40X5	6.5	71.51Ɛ	X.5	95X.X6
2.6	5.7X71	6.6	76.615	X.6	X08.1X
2.7	5.ƐƐ04	6.7	7Ɛ.X85	X.7	X7X.32
2.8	6.4241	6.8	85.716-	X.8	Ɛ35.60
2.9	6.8887	6.9	8Ɛ.775	X.9	ƐƐ6.1X
2.X	7.1639	6.X	96.051	X.X	1080.7
2.Ɛ	7.6741	6.Ɛ	X0.996-	X.Ɛ	1151.1
3.0	8.0000	7.0	X8.000	Ɛ.0	1228.0
3.1	8.5860	7.1	Ɛ3.740	Ɛ.1	1309.9
3.2	8.Ɛ90Ɛ	7.2	ƐƐ.813	Ɛ.2	13Ɛ6.X
3.3	9.61Ɛ7	7.3	108.27	Ɛ.3	14XƐ.6-
3.4	X.0Ɛ52	7.4	115.33	Ɛ.4	15Ɛ0.4
3.5	X.818X	7.5	122.X4	Ɛ.5	16Ɛ9.9
3.6	Ɛ.3921	7.6	131.03	Ɛ.6	1814.4
3.7	Ɛ.ƐX07	7.7	13Ɛ.95	Ɛ.7	1938.6
3.8	10.848	7.8	14Ɛ.23	Ɛ.8	1X6Ɛ.0
3.9	11.555	7.9	15Ɛ.33	Ɛ.9	1ƐƐ0.4
3.X	12.305	7.X	170.0X	Ɛ.X	2141.1
3.Ɛ	13.128	7.Ɛ	181.77	Ɛ.Ɛ	22X2.1
				10.0	2454.0

"The Curve $y = 2^x$", and the "Interval Chart" without once considering a unit of time. Instrument designers, piano tuners, and physicists are the ones who worry about absolute pitch. The most important parts of artistic music are the combinations and sequences of intervals and relative pitches. The whole theory and notation of intervals is complete with but one assumption, that of the octave.

Now to apply our notation to absolute pitch, we must make a second arbitrary assumption and choose a unit of time, thereby placing the limits of audible sound upon the curve. If we use a time unit of one second, the interval of *0.04623* (equals $\frac{\log 308}{\log 2} - 8.9$ duo) may be added to each exponent in the x (or notation) column of the "Duodecimal Equally Tempered Chromatic Scale" to find the notation for the Equally Tempered Chromatic Scale of American Standard Pitch A 440 (dec), *308* (duo). Then our notation for *308* (A 440) cycles per second would be *8.94623*. However, if a time unit other than the second is used, we will find it necessary to convert from one time system to another.

Let there be two systems in our notation, one based on time unit S, the other based on time unit T, where (T) (K) = S, K is a known constant factor between one unit time S and one unit time T. Suppose we have an absolute pitch (call it A) which is sounded by U cycles per time unit T. Then A will also be sounded by a frequency of (U)(K) cycles per time unit S. Now the notation for U cycles per T is $\frac{\log U}{\log 2}$, and the notation for U cycles per S is also $\frac{\log U}{\log 2}$, but the absolute pitches of these two notations are different due to the difference in the time units. It seems convenient to be able to convert the notation for absolute pitch A in time system T to the notation for the same pitch in time system S, by the addition or subtraction of a constant interval. This is handled by equation (5)

$$\frac{\log U}{\log 2} + \frac{\log K}{\log 2} = \frac{\log (UK)}{\log 2}; \quad \text{where } (TK) = S \qquad (5)$$

time unit T, notation for absolute pitch	conversion interval, constant distance between the two notations	time unit S, notation for the same absolute pitch

Let time unit S be one second, and let time unit T be $\frac{1}{2000}$ (duo) part of an hour, call it one threedovic (*30* Vics).

There are *2100* seconds in one hour, and *2000* threedovics in one hour. Therefore $T\left(\frac{20}{21}\right) = S$. Then suppose we have *194* cycles per threedovic; our notation for this is *8.0*. What should the notation be for the same pitch (*194* cycles per threedovic) in cycles per second? Substitute the above values in equation (5). *-0.085 928* is the conversion interval for all frequencies.

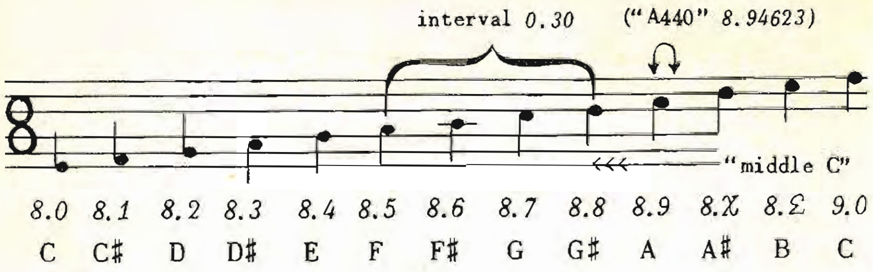
$$\left[\frac{\log 194}{\log 2} = 8.0\right] + \left[\frac{\log 20 - \log 21}{\log 2} = -0.085\ 928\right] = 7.\text{Ɛ}3629 \text{ (ans.)} \qquad (6)$$

It seems better to stick to the second, for a while at least, because time is already sufficiently dozenal for practical purposes, and to abolish it would raise a psychological barrier in inquiring minds of laymen. The dozenal quality of time is one of the few bridges which laymen may cross to duodecimals. It is a "foot in the door" so to speak. However, the choice of a unit of time is left to the reader.

However, we should aim for a duodecimal division of the day, just as we have the duodecimal division of the circle. The choice of a suitable fraction of the day as the unit of time for this musical notation is left to the reader.

SUMMARY Someone might suggest that it would be simpler to use the frequency numbers themselves, rather than all these logarithms and exponents. It would not be for three reasons. First, musical frequencies are usually handled by multiplication and division, whereas their logs (on base two calculated with the aid of base dozen), are added and subtracted. Second, interval distances between frequencies do not remain constant for one sort of sound sensation (octave, fifth), whereas the interval distances between the logarithms of the frequencies of a given ratio remain constant at every level of pitch (see "The Curve $Y = 2^x$"). Third, the Equally Tempered Chromatic Scale cannot be expressed in simple frequency numbers, whereas their logarithms may be expressed with two digit numbers for the first dozen octaves.

It might be well to mention a few advantages of our notation. It should be understood that this article was designed to simplify the notation of theoretical music. The musician will be concerned primarily with Godjevatz's Stave, the two digit notation of the Duodecimal Equally Tempered Chromatic Scale, and the tempered intervals *.70*, *.50*, *1.0*, etc.



However, those who wish to examine physical music more closely may do so without discarding the artistic music in "Notation Godjevatz" and "Notation More", and without encountering an impossible jumble of small letters, capital letters, numbers, symbols, notes, frequencies, ancient scales, ratios, sharps, flats, naturals, arbitrary constants, arbitrary divisions, conversion formulae, ambiguities, arbitrations, pitch standards, triple primes, subscripts, and Roman numerals. Even with this lavish abundance of symbols and subtilities, the person who enters the maze finds himself striving to gain a greater latitude of expression and accuracy. An Exponential Expression for Music offers this latitude of expression and accuracy with but one set of duodecimal numbers governed by one theoretical, simple, basic concept; $y = 2^x$.

BOOKS Those who wish to strike at the root of theoretical music, will find Helmholtz's work of great value.

Helmholtz, Hermann L. F., *Sensations of Tone,* trans. from the 4th, 1877, German Ed. by Ellis; (4th trans. ed. Longmans 1912).

Mr. Terry's *Duodecimal Arithmetic* (Longmans 1938) is an essential tool in the use of duodecimal mathematical functions. Regardless of the number base, his 9 place book of tables is more legible, further extended, more modern, better arranged, more complete, and more convenient than the great majority of decimal tables. In comparison, nearly all 4, 5, 6, and 7 place decimal tables may be termed mediocre. The Duodecimalists have a book of tables which surprisingly few Decimalists can match.

COMPLETE SOLUTION of THE PROBLEM OF THE SUM OF TWO SQUARES BY TRIGONOMETRY

by H. C. Robert, Jr.

The complete solution of that interesting theory of numbers problem $A^2 + B^2 = N = (P_1^r)(P_2^s)(P_3^t)$ . . . may be found by a simple trigonometric device. Since the method to be reported here can be derived from the solution obtained through the expansion of complex numbers by D. Chelini, (see Dickson's "History of the Theory of Numbers", vol. II, pg. 239), no derivation or proofs will be given. A description of the method of solution will serve to outline the interesting relationships involved.

If P is an odd prime of the form *10*p + *1* or *10*p + *5* (using dozenal notation,) its representation as the sum of two squares is said to be unique, that is, $P = a^2 + b^2$, in one way and one way only where (a, b) are integers with no common factor greater than 1. Now every prime, P, which can be so represented, can also be represented by one and only one angle, y, the tangent of which is, $\tan y = \frac{b}{a} = \frac{B}{A}$. Substituting in $A^2 + B^2 = N$, $P = a^2 + b^2$.

Now if $N = (P_1)(P_2)$ and y_1 and y_2 are the angles corresponding to P_1 and P_2 and their tangents are, $\tan y_1 = \frac{b}{a}$, $\tan y_2 = \frac{d}{c}$, first we determine the tangent of the sum of the angles, -

$$\tan (y_1 + y_2) = \frac{(b/a) + (d/c)}{1 - (b/a)(d/c)} = \frac{B_1}{A_2} = \frac{bc + ad}{ac - bd}$$

next we determine the tangent of the difference between the angles, -

$$\tan (y_1 - y_2) = \frac{(b/a) - (d/c)}{1 + (b/a)(d/c)} = \frac{B_2}{A_1} = \frac{bc - ad}{ac + bd}$$

Now we have $A_1^2 + B_1^2 = A_2^2 + B_2^2 = N = (P_1)(P_2)$, and substituting we obtain $(ac - bd)^2 + (bc + ad)^2 = (ac + bd)^2 + (bc - ad)^2 =$

$$= (a^2 + b^2)(c^2 + d^2)$$

Thus by simply finding the tangents of the sum and difference of two angles, we have arrived at the familiar four parameter identity for two sums of two squares. If one of the two primes forming our composite, say P_2, is equal to 2, then $d = c = 1$, and instead of having two representations for such a composite, we will have only one. That is, if $a^2 + b^2 = P_1$, then $(a - b)^2 + (a + b)^2 = 2P_1$. The only new factor involved is in our trigonometric approach, where we find that we multiply by two by adding or subtracting the angle whose tangent is 1 to the angle representing the multiplicand.

We may extend this method to $N = (P_1)(P_2)(P_3)$, by adding or subtracting the angle, y_3, which represents P_3, to the results of the case just considered. Let $\tan y_3 = \frac{f}{e}$, then we find,–

$$\tan (y_1 + y_2 + y_3) = \frac{B_1}{A_1} = \frac{bce + ade + acf - bdf}{ace - bde - bcf - adf}$$

$$\tan (y_1 + y_2 - y_3) = \frac{B_2}{A_2} = \frac{bce + ade - acf + bdf}{ace - bde + bcf + adf}$$

$$\tan (y_1 - y_2 + y_3) = \frac{B_3}{A_3} = \frac{bce - ade + acf + bdf}{ace + bde - bcf + adf}$$

$$\tan (y_1 - y_2 - y_3) = \frac{B_4}{A_4} = \frac{bce - ade - acf - bdf}{ace + bde + bcf - adf}$$

Now substituting in, – $A_1^2 + B_1^2 = A_2^2 + B_2^2 = A_3^2 + B_3^2 = A_4^2 + B_4^2 = N$

we obtain,– $(bce + ade + acf - bdf)^2 + (ace - bde - bcf - adf)^2 =$

$= (bce + ade - acf + bdf)^2 + (ace - bde + bcf + adf)^2 =$

$= (bce - ade + acf + bdf)^2 + (ace + bde - bcf + adf)^2 =$

$= (bce - ade - acf - bdf)^2 + (ace + bde + bcf - adf)^2 =$

$= (a^2 + b^2)\ (c^2 + d^2)\ (e^2 + f^2)$

And we have a six parameter solution for the identity of four sets of the sums of two squares. In the same manner we may extend this solution to composites with four or more factors. It should be noted that in the foregoing two cases we have used only half of the total number of variations of signs in combining the angles, y_1, y_2, y_3, etc. This is because the other possible combinations of signs may be considered as complements of the combinations we have used. Thus + + + is the complement of – – – ; + + – of – – + ; + – + of – + – and + – – of – + + . We have used the first in each case. The complement only changes the sign of B, and upon squaring the result is identical with that obtained through use of the other combination. For some purposes all combinations may be required.

Numerical example of the six parameter solution:

Let $a = 2$; $b = 1$; $c = 3$; $d = 2$; $e = 4$; $f = 1$; and we obtain

(Base X)

$33^2 + 4^2 = 32^2 + 9^2 = 31^2 + (-12)^2 = 24^2 + 23^2 = 1105 = 5.13.17$

(Base XII)

$29^2 + 4^2 = 28^2 + 9^2 = 27^2 + (-10)^2 = 20^2 + 1Ɛ^2 = 781 = 5.11.15$

So far we have only considered composites made up of the first powers of primes. We should now consider the representations of higher powers of primes as the sum of two squares. Let the angle, y, the tangent of which is $\tan y = \frac{b}{a}$, represent a prime, P. We determine, -

$$\tan (y + y) = \frac{(b/a) + (b/a)}{1 - (b/a)(b/a)} = \frac{B_1}{A_1} = \frac{2ab}{a^2 - b^2}$$

$$\tan (y - y) = \frac{(b/a) - (b/a)}{1 + (b/a)(b/a)} = \frac{B_2}{A_2} = \frac{o}{a^2 + b^2}$$

Now substituting in $A_1^2 + B_1^2 = A_2^2 + B_2^2 = P^2$

we obtain $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2 + o^2 = P^2$

This last expression is obviously our familiar two parameter solution for the Pythagorean Triangle. Nothing new, but the generators, (a, b), instead of being figures pulled out of the air have an added dignity. They define the tangent of an angle, an angle that is exactly half of one of the angles of the triangle which they generate.

A complete investigation of the trigonometric meaning of these generators is beyond the scope of this introductory paper but we should note several interesting relationships. If (a, b) are relatively prime, one even and the other odd, the triangle which they generate will be "primitive". Obviously all multiples of this triangle will have the same angles even though the generators appear to change. Now if we multiply both numerator and denominator of tan y by $\sqrt{k}$, thus, -

$$\tan y = \frac{b}{a} = \frac{b\sqrt{k}}{a\sqrt{k}}$$

and let $k = 1, 2, 3, 4, \ldots\ldots.$, we obtain all multiples of the "primitive" triangle. Now we also know that there are two acute angles to the triangle and if it is generated by the tangent of half of one angle, then what of the other angle? Let the two half angles be y_A and y_B. We know the sum of the half angles is half of one right angle, that is, is the angle whose tangent is 1. Thus if

$$\tan y_A = \frac{b}{a}, \text{ we may write}$$

$$\tan y_B = \frac{1 - b/a}{1 + b/a} = \frac{a - b}{a + b}$$

and as we have shown earlier, if $a^2 + b^2 = P$, then $(a - b)^2 + (a + b)^2 = 2P$, and using $(a + b,\ a - b)$ as generators we obtain the two-multiple of the primitive triangle generated by (a, b). From this relationship we find that all angles with rational tangents occur in pairs where $\tan y_A = \frac{b}{a}$, with either a or b even, the other odd, and $\tan y_B = \frac{a - b}{a + b}$, with both numerator and denominator odd.

Thus every angle with a rational tangent is a member of such a pair, and every such pair determines a primitive Pythagorean triangle and all of its multiples. Note that the triangle is actually determined by the pair although only one angle is used. Before leaving this subject we might also note, since all trigonometric functions of a Pythagorean triangle are rational, that, if y is any angle with a rational tangent, then all functions of $(2y)$ are rational.

Proceeding to the next step, $A^2 + B^2 = P^3$, we determine, -

$$\tan (2y + y) = \frac{B_1}{A_1} = \frac{3a^2b - b^3}{a^3 - 3ab^2}$$

$$\tan (2y - y) = \frac{B_2}{A_2} = \frac{a^2b + b^3}{a^3 + ab^2}$$

Now substituting in $A_1^2 + B_1^2 = A_2^2 + B_2^2 = N = P^3$

we obtain $(a^3 - 3ab^2)^2 + (3a^2b - b^3)^2 = (a^3 + ab^2)^2 +$

$$+ (a^2b + b^3)^2 = (a^2 + b^2)^3 = P^3$$

For a numerical example for this last case, let $a = 2$; $b = 1$ and we obtain (Base X), - $11^2 + 2^2 = 10^2 + 5^2 = 5^3 = 125$ or for Base XII, - $\mathcal{E}^2 + 2^2 = \mathcal{X}^2 + 5^2 = 5^3 = \mathcal{X}5$

This method may be continued to obtain two parameter solutions for the partitioning of any power of P into two squares.

We may state the general solution for $A^2 + B^2 = P^r$ as follows:

Determine all values of $\tan Z = \frac{B}{A}$

where $Z = (r - n)\,y - ny$

and y is the angle which represents the prime P.

and $n = 0, 1, 2, \ldots\ r$, if all representations are required,

or $N \leqq (r/2)$ if difference in sign of quantities to be squared can be ignored.

After obtaining all values of tan Z, we take the sum of the squares of the numerator and denominator of each value and this sum forms one member of the identity, which will have as many members as there are values of tan Z. Care must be taken not to simplify the values of the tangent since it will be noted that when n has any value other than zero, the numerator and denominator can both be divided by P.

The results for P_1^r, P_2^s, P_3^t, etc., may be combined exactly as for composites of the first powers by combining the angles Z_1, Z_2, Z_3, etc. Thus we may solve, for example.- $A^2 + B^2 = N = (P_1^3)(P_2^2)$. Let $P_1 = a^2 + b^2$, $\tan y_1 = \frac{b}{a}$, and $P_2 = c^2 + d^2$, $\tan y_2 = \frac{d}{c}$, then we find that

$$A_1^2 + B_1^2 = A_2^2 + B_2^2 = A_3^2 + B_3^2 = A_4^2 + B_4^2 = A_5^2 + B_5^2 =$$

$$= A_6^2 + B_6^2 = (a^2 + b^2)^3(c^2 + d^2)^2 \text{ is solved by,-}$$

$$A_1 = a^3c^2 - 3ab^2c^2 - a^3d^2 + 3ab^2d^2 - 6a^2bcd + 2b^3cd$$

$$B_1 = 3a^2bc^2 - b^3c^2 - 3a^2bd^2 + b^3d^2 + 2a^3cd - 6ab^2cd$$

$$A_2 = a^3c^2 - 3ab^2c^2 - a^3d^2 + 3ab^2d^2 + 6a^2bcd - 2b^3cd$$

$$B_2 = 3a^2bc^2 - b^3c^2 - 3a^2bd^2 + b^3d^2 - 2a^3cd + 6ab^2cd$$

$$A_3 = a^3c^2 - 3ab^2c^2 + a^3d^2 - 3ab^2d^2$$

$$B_3 = 3a^2bc^2 - b^3c^2 + 3a^2bd^2 - b^3d^2$$

$$A_4 = a^3c^2 + ab^2c^2 - a^3d^2 - ab^2d^2 - 2a^2bcd - 2b^3cd$$

$$B_4 = a^2bc^2 + b^3c^2 - a^2bd^2 - b^3d^2 + 2a^3cd + 2ab^2cd$$

$$A_5 = a^3c^2 + ab^2c^2 - a^3d^2 - ab^2d^2 + 2a^2bcd + 2b^3cd$$

$$B_5 = a^2bc^2 + b^3c^2 - a^2bd^2 - b^3d^2 - 2a^3cd - 2ab^2cd$$

$$A_6 = a^3c^2 + ab^2c^2 + a^3d^2 + ab^2d^2$$

$$B_6 = a^2bc^2 + b^3c^2 + a^2bd^2 + b^3d^2$$

For a numerical example, let $a = 2$; $b = 1$; $c = 3$; $d = 2$; and we have Base X,-

$$(-122)^2 + 79^2 = 142^2 + 31^2 = 26^2 + 143^2 =$$

$$= (-10)^2 + 145^2 = 110^2 + (-95)^2 = 130^2 + 65^2 = (5^3)(13^2) =$$

$$= 21125$$

Base XII

$$(-\mathcal{X}2)^2 + 67^2 = \mathcal{E}\mathcal{X}^2 + 27^2 = 22^2 + \mathcal{E}\mathcal{E}^2 =$$

$$= (-\mathcal{X})^2 + 101^2 = 92^2 + (-7\mathcal{E})^2 = \mathcal{X}\mathcal{X}^2 + 55^2 = (5^3)(11^2) =$$

$$= 10285$$

Using methods identical with the foregoing we may also solve the equation $A^2 - B^2 = N$, where every prime, P, may be represented by the tan $y = \frac{b}{a}$, and y_1, y_2, y_3, etc., are combined as usual for obtaining the hyperbolic tangent of the sum and differences of quantities. In this problem it is obvious that $P = a^2 - b^2$.

The duplication of a complete solution by other methods is of little or no importance. It may be important that we have found some interesting relationships between trigonometry and number theory that have been neglected since the days of the Babylonians who used tables of Pythagorean Triangles to measure the trigonometric functions. A thorough investigation of the trigonometry of number theory may be a fertile field for extension of our knowledge.

MATHEMATICAL RECREATIONS

D. M. Brown, Editor

In the last issue we requested suggestions for new kinds of recreational material. So far the suggestion box is sadly neglected. However, several errors have been called to our attention, and we are glad to get *some* mail.

In the June, 1949 issue of this publication, the answer to problem 1 should be $x = 1$, $y = 6$. In problems 3 and 4, the last plus sign should be replaced by an equality sign.

Lewis C. Seelbach submitted a "Lazy" solution to the magic square given on page *15* of the last issue. The solution is

L A Z Y C O M P U T E R
0 1 2 3 4 5 6 7 8 9 X Ɛ

There is enough information given in the problem as stated to solve it by use of algebra. We'll be glad to publish an algebraic solution.

To emphasize the value of various bases in easy solution to certain types of problems, the following problems are submitted:

1. The weight problem.

A merchant has a scale for weighing articles consisting of two balanced pans. He desires to be able to weigh articles weighing from *1* to *100* lbs. What is the most economical set of weights to use? What number-base is involved?

2. The coin problem.

The same merchant is aware that of N coins of the same denomination, *one* is spurious, and of different weight than the others. Using the balance and the coins, how many weighings must he be permitted to make to determine the following:-

a. Which coin is spurious, if he knows that the spurious coin is heavier (or lighter) than the others?
b. Which coin is spurious if he knows only that its weight is different from the others?
c. Which coin is spurious, and which is heavier, the spurious coin or a good one, if he knows only that the good and spurious coins have different weights?
d. How many weighings are permitted if N = *3, 4, 5, 6,* .... etc?
e. What number base is most convenient for the problem?

HALVING

One of the simple ways of dividing things into small parts is to successively divide them into halves or thirds, again and again, until the desired fraction is reached. Man has frequently shown his preference for arranging the scales of his standards of weight and measure to facilitate this type of subdivision. Familiar illustrations are the foot with twelve inches, and the pounds of twelve and sixteen ounces. In analysing factors of number bases for general use, such divisibility is one of the important criteria. The following table will afford a visual comparison of the degree of facility of successive halving on different bases.

Fractions	*Fractionals*			
	Base Two	*Base Eight*	*Base Ten*	*Base Twelve*
1/2	.1	.4	.5	.6
1/4	.01	.2	.25	.3
1/8	.001	.1	.125	.16
1/16	.000 1	.04	.062 5	.09
1/32	.000 01	.02	.031 25	.046
1/64	.000 001	.01	.015 625	.023
1/128	.000 000 1	.004	.007 812 5	.011 6
1/256	.000 000 01	.002	.003 906 25	.006 9
1/512	.000 000 001	.001	.001 953 125	.003 46

It is surprising that the twelve base should approach the octic base as closely as it does in the ease of accommodating this operation. The ten base requires exactly as many places in each step as does the binary base, - the twelve base half, and the octic base a third as many places.

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AN ALPHABETICALCULATION

by Philip Haendiges

A	Establishes	Justifiably	Of
Bright	Figures	Killing	Problematical
Calculator	Giving	Long	Quantities.
Duodecimally	Him	Mathematical	
	Information	Notations	

Result:	Saving	With
	Time,	X-ellent
	Understanding	Yardsticks
	Values,	Zero-iferous.

PYTHAGOREAN TRIANGLES and THEIR INSCRIBED CIRCLES

by H. C. Robert, Jr.

In the Bulletin of October 1948 (Vol. 4, No. 2, page 15), a brief discussion of Pythagorean Triangles with equal perimeters was presented. An even more neglected subject is the relationship between Pythagorean Triangles and the circles inscribed in them.

The relationship, $A^2 + B^2 = C^2$, is usually derived from the generators, (m, n) where $A = m^2 - n^2$, $B = 2mn$, $C = m^2 + n^2$. In terms of these generators, the radius, R, of the circle inscribed in a Pythagorean Triangle is $R = n\ (m - n)$.

For $R = 1$, it is obvious that $n = 1$ and $(m - n) = 1$, or $m = 2$. These generators produce our smallest Pythagorean Triangle, the *3, 4, 5,* triangle of Figure 1. Since the sides, *3, 4, 5,* have no common factor, this triangle is called a "primitive" triangle. Such triangles are produced if, and only if, the generators (m, n) are relatively prime and one of them is even. No other right triangle with integral sides can be circumscribed about the circle of unit radius. Obviously a multiple of the *3, 4, 5,* triangle can be circumscribed about every circle with an integral radius. Note how the foregoing corresponds *exactly* with the characteristics of unity, that is, the only integral factor of unity is unity itself and unity is a factor of any integer.

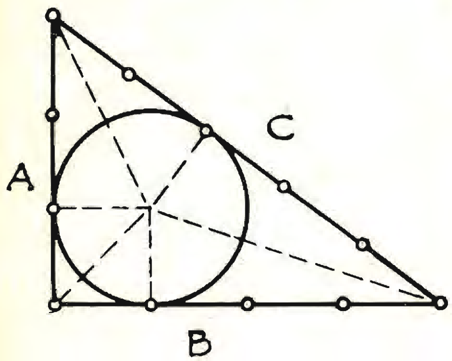

FIG. 1

Before passing on to other triangles, it is interesting to note some of the relationships between a Pythagorean Triangle and its inscribed circle:

1. The tangents from the vertices to the inscribed circle are necessarily integral.

2. The tangent of half of the vertex angles is rational. All functions of the whole angles are rational.
3. The excess of the sum of the two legs of the triangle over the hypotenuse is equal to twice the radius of the inscribed circle.
4. The area of the triangle is the product of half its perimeter and the radius of the inscribed circle. The half-perimeter can also be represented as the sum of the tangent distances from the three vertices to the inscribed circle.

Now let us investigate the case of the circle with a radius of 2. We can circumscribe two and only two right triangles about this circle:

1. $(3, 2)$ A = 5, B = 10, C = 11
2. $(3, 1)$ A = 8, B = 6, C = $\mathcal{X}$

The first of these, the 5, 10, 11, triangle is primitive and a multiple of this triangle can be circumscribed about every circle with an even radius. The second triangle is not primitive, being the 2-multiple of the 3, 4, 5, triangle for the unit circle. No other integral right triangles can be circumscribed about the circle with radius, R = 2.

Now for R = p, where p is any odd prime, we find that three and only three triangles may be circumscribed about such a circle, thus:

1. $(p + 1, p)$ A = $2p + 1$, B = $2p^2 + 2p$, C = $2p^2 + 2p + 1$
2. $(p + 1, 1)$ A = $p^2 + 2p$, B = $2p + 2$, C = $p^2 + 2p + 2$
3. $(2\sqrt{p}, \sqrt{p})$ A = $3p$, B = $4p$, C = $5p$

The first two triangles are primitive, the third is the p-multiple of the 3, 4, 5, triangle for the unit circle. Actually the first of these triangles is primitive not only for prime values of p but for every integral value of p, even or odd. It is the well known case for C - B = 1, the solution of which is generally credited to Pythagoras. The second triangle is also a well known case, that of C - A = 2, and this triangle will be primitive for all odd values of p whether prime or composite. For cases of even p, this triangle will be the 2-multiple of the first triangle as may be verified by substituting $2p$ for p in the second triangle and comparing the result with the first triangle multiplied by 2. Multiples of these two primitive triangles for each prime, (or odd composite) can be circumscribed about every circle, the radius of which is a multiple of p.

If R = ab, where a and b are two distinct odd primes greater than 1, we get the three same triangles, two primitive and one the p-multiple of 3, 4, 5, as given previously for a prime radius. It is only necessary to substitute $p = ab$ in the forms given. We also get the a-multiples of the two primitive triangles belonging to the prime, b, thus: A = $2ab + a$, B = $2ab^2 + 2ab$, C = $2ab^2 + 2ab + a$, and A = $ab^2 + 2ab$, B = $2ab + 2a$, C = $ab^2 + 2ab + 2a$ and we get the b-multiples of the two primitive triangles belonging to the prime, a, thus:

A = $2ab + b$, B = $2a^2b + 2ab$, C = $2a^2b + 2ab + b$
and A = $a^2b + 2ab$, B = $2ab + 2b$, C = $a^2b + 2ab + 2b$

and in addition we get two new primitive triangles, thus:

$(a + b, a)$ A = $2ab + b^2$, B = $2a^2 + 2ab$, C = $2a^2 + 2ab + b^2$
$(a + b, b)$ A = $a^2 + 2ab$, B = $2ab + 2b^2$, C = $a^2 + 2ab + 2b^2$

Thus for an odd composite with two distinct factors, we have the same three triangles that would characterize a prime, and in addition we have six other triangles, four non-primitive and two primitive. It is obvious from the distribution of the two factors in the several members of these six additional triangles, that if any one of these triangles is known, we can find the factors of the radius R = ab. Unfortunately, we know of no way of finding any one of these triangles unless we first know the factors of R. Otherwise we would have a solution to the eternal problem of factorisation.

When the radius, R, is even, there are two or more primitive triangles except when R is a power of 2, for which case there is only one primitive triangle. The number of non-primitive triangles is a rather complicated function of the factors of the radius, R, but the number of primitive Pythagorean Triangles, PT, which can be circumscribed about any given radius, R, can be simply stated, thus: For R = $2^a f_1^b f_2^c f_3^d \ldots f_n^z$ where $f_1, f_2, \ldots f_n$ are n distinct odd primes greater than 1, the number of primitive triangles, PT, is: PT = 2^n. Thus when R = 1 or R = 2^k, we will have $n = 0$ and PT = $2^0 = 1$ and when R = p, p being an odd prime, $n = 1$ and PT = 2, as has been stated previously.

Since there is at least one, and generally two or more primitive Pythagorean Triangles for every integral value of R, it appears that there are many more such triangles than there are numbers. For example, there are 278 primitive Pythagorean Triangles for which the radius of the inscribed circles does not exceed 100.

All Pythagorean Triangles which may be circumscribed about a circle of radius, R, may be found from the factorisations of R = $k_1a_1b_1 = k_2a_2b_2 = k_3a_3b_3 = k_na_nb_n$ where k_1, k_2, etc., either equal *1*, or are odd numbers without square factors. There will be as many circumscribed triangles as there are different factorisations including reversals of order of the factors a and b. The triangles will be primitive if, and only if, $k = 1$ and a and b are relatively prime and b is odd. The generators from which the sides may be found are obtained from these factorisations of R by means of:

$$m = (a + b)\sqrt{k} \quad \text{and} \quad n = a\sqrt{k}$$

The resemblance between the foregoing and the method of handling perimeter problems given in the Bulletin of October 1948 is obvious.

This brief discussion does little more than scratch the surface of the neglected subject of the relationships between Pythagorean Triangles, the circles inscribed in them, the tangent distances and other items that are introduced when we consider the inscribed circles. Further investigation of the subject may be both interesting and profitable. A table giving all triangles with a radius of less than *16* follows, arranged according to radius and perimeter. Many obvious and unexpected relationships can be noted in this tabulation.

SOLUTIONS OF PYTHAGOREAN TRIANGLES
ARRANGED ACCORDING TO RADII OF THE INSCRIBED CIRCLES

Radius	Generators		A	B	C	Perimeter
1	2, 1	(p)	3	4	5	10
2	3, 2	(p)	5	10	11	26
	3, 1		8	6	X	20
3	4, 3	(p)	7	20	21	48
	4, 1	(p)	13	8	15	34
	$2\sqrt{3}$, $\sqrt{3}$		9	10	13	30
4	5, 4	(p)	9	34	35	76
	5, 1		20	X	22	50
	4, 2		10	14	18	40
5	6, 5	(p)	Ɛ	50	51	Ɛ0
	6, 1	(p)	2Ɛ	10	31	70
	$2\sqrt{5}$, $\sqrt{5}$		13	18	21	50

Radius	Generators		A	B	C	Perimeter
6	7, 6	(p)	11	70	71	132
	7, 1		40	12	42	94
	$3\sqrt{3}$, $2\sqrt{3}$		13	30	33	76
	5, 3		14	26	2X	68
	$3\sqrt{3}$, $\sqrt{3}$		20	16	26	60
	5, 2	(p)	19	18	25	5X
7	8, 7	(p)	13	94	95	180
	8, 1	(p)	53	14	55	100
	$2\sqrt{7}$, $\sqrt{7}$		19	24	2Ɛ	70
8	9, 8	(p)	15	100	101	216
	9, 1		68	16	6X	130
	6, 4		18	40	44	X0
	6, 2		28	20	34	80
9	X, 9	(p)	17	130	131	278
	X, 1	(p)	83	18	85	164
	$4\sqrt{3}$, $3\sqrt{3}$		19	60	63	120
	$4\sqrt{3}$, $\sqrt{3}$		39	20	43	X0
	6, 3		23	30	39	90
X	Ɛ, X	(p)	19	164	165	326
	Ɛ, 1		X0	1X	X2	1X0
	7, 5		20	5X	62	120
	$3\sqrt{5}$, $2\sqrt{5}$		21	50	55	106
	7, 2	(p)	39	24	45	X6
	$3\sqrt{5}$, $\sqrt{5}$		34	26	42	X0
Ɛ	10, Ɛ	(p)	1Ɛ	1X0	1X1	3X0
	10, 1	(p)	ƐƐ	20	101	220
	$2\sqrt{Ɛ}$, $\sqrt{Ɛ}$		29	38	47	Ɛ0
10	11, 10	(p)	21	220	221	462
	11, 1		120	22	122	264
	$5\sqrt{3}$, $4\sqrt{3}$		23	X0	X3	1X6
	8, 6		24	80	84	168
	$5\sqrt{3}$, $\sqrt{3}$		60	26	66	130
	8, 2		50	28	58	114
	7, 4	(p)	29	48	55	10X
	$4\sqrt{3}$, $2\sqrt{3}$		30	40	50	100
	7, 3		34	36	4X	Ɛ8
11	12, 11	(p)	23	264	265	530
	12, 1	(p)	143	24	145	2Ɛ0
	$2\sqrt{11}$, $\sqrt{11}$		33	44	55	110

Radius	Generators	A	B	C	Perimeter
12	13, 12 (p)	25	2Ɛ0	2Ɛ1	606
	13, 1	168	26	16X	340
	9, 7	28	X6	XX	200
	$3\sqrt{7}$, $2\sqrt{7}$	2Ɛ	70	77	156
	9, 2 (p)	65	30	71	146
	$3\sqrt{7}$, $\sqrt{7}$	48	36	5X	120
13	14, 13 (p)	27	340	341	6X8
	14, 1 (p)	193	28	195	394
	$6\sqrt{3}$, $5\sqrt{3}$	29	130	133	290
	$4\sqrt{5}$, $3\sqrt{5}$	2Ɛ	X9	X5	1Ɛ4
	$6\sqrt{3}$, $\sqrt{3}$	89	30	93	190
	8, 5 (p)	33	68	75	154
	$4\sqrt{5}$, $\sqrt{5}$	63	34	71	148
	$2\sqrt{13}$, $\sqrt{13}$	39	50	63	130
	8, 3 (p)	47	40	61	128
14	15, 14 (p)	29	394	395	796
	15, 1	200	2X	202	430
	X, 8	30	114	118	260
	X, 2	80	34	88	180
	8, 4	40	54	68	140
15	16, 15 (p)	2Ɛ	430	431	890
	16, 1 (p)	22Ɛ	30	231	490
	$2\sqrt{15}$, $\sqrt{15}$	43	58	71	150

NOTE: (p) following the generators indicates a Primitive Triangle.

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EXTRA COPIES

Many of those interested in duodecimals sedulously maintain complete files of the Duodecimal Bulletin. This is a practice which we wish to encourage. These esteemed people are sometimes confronted with the need for cutting out some table or article from the Bulletin for special use. On request, we will gladly supply extra copies, so that they can clip the needed material, yet maintain their Bulletin files in proper condition.

THE MAIL BAG

Sons! Sons! Sons! Everybody is having boy babies. If we don't be getting some duodecimal daughters pretty soon, what are we going to do about double-duodecimal grandchildren?

Mr. and Mrs. Jamison Handy, Jr., announce the debut of Galen William Handy. Two of our Canadian families, Mr. and Mrs. Leon L'Heureux, and Mr. and Mrs. Edwin Bobyn jubilate over sons. It is the second son for the L'Heureuses. Paul and Cam Adams (the Mad Adamses) have a son. Bob and Mary Lloyd have their third son. Ain't nobody got a daughter? Something will have to be done about this!

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We are delighted to present Trenchard More's article on An Exponential Expression for Music. This scholarly paper supplements beautifully the earlier work of Velizar Godjevatz on the New Musical Notation which appeared in the Bulletin for October, 1948.

Not only is Trenchard More to be congratulated on a fine piece of work, but we are to be congratulated on having among the rising generation so capable an advocate of duodecimals. We expect to hear the reverberations of the note these pioneers have struck.

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We are eager to find among our members a source for a paper on a duodecimal color notation. The Munsell decimal color notation is important commercially, but it suffers from some distortion in being compressed within the decimal limitation. A duodecimal notation would be a definite refinement, and would require but minor modification of the Munsell System.

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Louis Paul d'Autremont's article on the Duodecimal Perpetual Calendar has had a fine reception. In the Havana daily Informacion, Juan de Dios Tejada commented favorably and at considerable length on the d'Autremont proposal in his column, La Marcha de la Technica.

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The Board of Directors of the Society has recently authorized public announcement of our willingness to furnish without charge sets of introductory duodecimal literature to the pupils of mathematics classes of teachers colleges, to the extent that our supply permits. Sets will comprise a copy of the Duodecimal Bulletin, a copy of the reprint of the Excursion in Numbers, by F. Emerson Andrews, and a copy of the Society's folder.

Ye Ed.