The Duodecimal Bulletin

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THE DUODECIMAL SOCIETY OF AMERICA

20 Carlton Place ~ ~ ~ ~ ~ Staten Island 4, N. Y.

THE DUODECIMAL SOCIETY OF AMERICA

is a voluntary nonprofit organization for the conduct of research and education of the public in the use of Base Twelve in numeration, mathematics, weights and measures, and other branches of pure and applied science.

Full membership with voting privileges requires the passing of elementary tests in the performance of twelve-base arithmetic. The lessons and examinations are free to those whose entrance applications are accepted. Remittance of $6, covering initiation fee ($3) and one year's dues ($3), must accompany applications.

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The Duodecimal Bulletin

SYNOPSIS OF THE TWECIMAL SYSTEM

by Alfred Norland, C. E.

The coming of this new Aquarian Age will bring great changes in our accustomed usages and living conditions. Signs of these changes are evident in all directions, - in the trend toward revaluation, improvement, and simplification. One field where marked simplification and improvement may be expected is in our numbers and measures.

For more than twenty years, my mind has been directed on the possibility of constructing a better practical system in which to handle our dimensions with ease. The lack of system in our existing measures is obnoxious and wasteful. Many efforts have been made to bring order to this chaos, but they have been little noted and soon forgotten.

The only system that has attained wide recognition is the French Decimal System of Weights and Measures. Tho not quite correctly based, its ordered structure has offered advantages in scientific use, and its basic principles have found general application in the young and growing field of electrical science and industry. Its progress into general use has been relatively slow because of the unpractical size of its units and because of the difficulties in effecting accurate conversions from existing popular measures.

Shortly after her revolution, France made the use of the decimal metric system compulsory. The rapidly growing trade between the nations of Central Europe in the latter part of the 19th Century led them, one after another, to adopt this system. England, alone, holds to its old units of currency and measure. The United States has found it advisable to follow England in this respect, and, since England and the United States are together preponderant in international trade, we find ourselves burdened with different sets of standards that are without system. Tho we suffer many inconveniences thru this lack of system, the sizes of our units of measure seem to suit us. We contrive means to offset the disadvantages.

In considering the requirements of a system that is to be superior to the metric, peculiar difficulties, and many paradoxical incongruities present themselves. The use of twelve as the scale of division of the units of weight and measure is much older than the use of ten for this purpose. Our measures of angle and of time are hoary with age, and any decimal division of these units fails lamentably of convenience. Only divisions by twelve seem to fall into a practical pattern. Many odd and seemingly unrelated things emphasize the ages-old preference for twelve. The year has twelve months. The Nordic languages have twelve simple independent counting words. Israel had twelve tribes. Jesus, the Christ, selected twelve apostles, and, losing one, made an extraordinary effort to appoint the twelfth, - the unit of twelve seeming to have some meaning to him. Tradition and practice seem to favor the duodecimal system.

Linguistically analyzed, the word, "duodecimal," does not well convey the accepted meaning of a twelve-system. First, "duo" means two, and "decimal" has nothing to do with twelve at all, but means ten. Their combination would be better applied to a system of twenties, rather than of twelves.

To me, the use of the syllable "twe," as an abbreviation for twelve, seems preferable. And the terms, "Twe System," and "Twecimals," are more expressive and proper. From this root-form, are derived the necessary number names, such as "twe-one" for *11*, "threetwe-eight" for *38*, etc. The word "gross" requires no change, but for *1 000*, *1 000 000*, and *1 000 000 000*, the words "threeos," "sixos," and "nineos," talk for themselves and are fitting correlatives for thousand, million, and billion.

For the new symbols necessary for ten and eleven, we should follow established precedent by inverting the three for ten, and the seven for eleven, as: *1 2 3 4 5 6 7 8 9 Ɛ L 10*. The names for these terms require no change, and *8Ɛ* and *ƐL* would naturally be called "eighttwe-ten," and "tentwe-eleven." Further eleboration here seems unnecessary.

The movement of the great hands of God's clock in the sky determine time regardless of what measures we use. Because of man's primitive dependence upon sundials and gnomons, he was unduly impressed with the daylight hours, and he early separated day from night, subdividing both by twelve. This twenty-four hour day-and-night we shall call the "dayn," but we will divide it by twelve. It is strange that we should have retained this archaic 24-hour measure when, among all instruments, watches and clocks are the most common.

God's clock runs at half the speed of man's. For the Twe System, we shall arrange our clocks in the same rational way, to operate at half speed. The A.M. hours will be those on the right half of the dial, and the P.M. hours will be those on the left.

The hands of our new clocks will correspond with the hour, minute and second hands of present clocks. The hour hand will rotate in one dayn, and the numerals of the dial will mark the twelve "stunds." The minute hand will rotate in one stund, and the numerals will count the twelve "whiles." The second hand will rotate in one while, the numerals marking the "minutes," and the scale divisions between numerals marking the twelve "moments" and their fractions. The lengths of these intervals are: the stund, two hours; the while, ten minutes; the new minute, fifty seconds; the moment, four and a sixth seconds. Thus, a reading of *4;256* (literally, 4 stunds, 2 twe 5 minutes and 6 moments,) would correspond exactly with 8:24' 35" A.M.

Measures of space depend upon the establishment of a standard of reference, such as the metal bars that are the prototypes of the standard meter, or some physical constant. Since the standard should be invariable, the velocity of light in vacuum has been chosen as the constant for the basis of the twecimal measures.

Astronomers use this constant to measure the dimensions of the universe, and we can bring it down to earth. Two hundred years ago, the Danish astronomer Romer measured the difference between the actual and the apparent occulation by Jupiter of one of its moons, and thus determined the velocity of light in space. Considering the crudity of his instruments, his accuracy was remarkable. The latest determination of this constant was made by the late Prof. A. A. Michelson, who found it to be 186 284 miles, or 299 796 kilometers per second, and stated this measurement to be exact within 10 kilometers either way.

This figure equals 776 187.5 miles per moment. We will let this distance represent *1 000 000 000* of our basic unit of length, which we will call the "link." The length of the link is therefore *9.6* inches, and the "tum," which is *.1* link, will be *.96* of an inch. Thus, *5* links nearly equal *4* feet, and *5* tums approximate *4* inches in length. *4* threeoslinks are a little more than a mile, and *1* fouroslink is three and one eight miles, or a little more than a league.

The measures of area are based on the square link which is called the "plate." The cubic measures follow the same pattern, in the "kubtum," and the "kublink." The volume of *100* kubtums equals *1* "kan," and the weight of this amount of water is the "kublod." More simply: *.1* kublink = *1* kan = *1* kublod.

The kublink approximates half a cubic foot, or 14.19 cu. dm. The kan approximates the imperial quart, and the kublod 2.6 pounds, avoirdupois, or 1.18 kilograms. The kubtum of water weighs one twelod, which is approximately equal to 8.2 grams, or .29 ounce.

The twecimal system offers a most convenient basis for currency. The dollar will naturally have *100* pennies, *10* pennies making one twe, and *10* twes one dollar. For the British the Twe System will free them from much of the work now involved in currency computations. It is proposed that there be three new coins, worth, respectively, *3* shillings, *3* pence and *1* farthing, and called the Brit, the Twe and the Farthing. Thus, there would be *10* farthings to the twe, and *10* twes to the Brit.

For the angular measure, the 90° quadrant is subdivided twecimally, which accommodates all multiples of angles of 15° and the 32 points of the compass most conveniently. The twecimal thermometer has *100* divisions between the boiling and the freezing points of water.

The energy measures formed from the various combinations of the primary measures of distance, weight and time of the twecimal system, facilitate the convenient transformations which the scientific application of the metric system has proved to be essential.

The twecimal system combines the mathematical advantages of the twelve-base with man's preference for the scale of twelve in the division of his weights and measures. This inter-related system of standards is directly designed to bring order to the present chaos of standards of weight and measure.

This brief presentation has abridged the details of the tables of the various measures and their equivalents. Those who are interested further will find them amply covered in my book on "The Twe System," which will be found in the files of most large libraries.

KIN OF THE GOLDEN MEAN*

by Col. Robert S. Beard

The golden rectangle or golden section, the five pointed star or pentacle, the golden mean, and the Fibonacci series of numbers, comprise such an interesting mathematical family that a study of its beauty and dynamic symmetry may cause one to take an extra puff on his pipe and indulge in a bit of imaginative thinking.

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The Golden Rectangle

The ancient Grecians considered that a rectangle possessed the most graceful form when the ratio of its width to its length was the same as the ratio of its length to the sum of its width and its length. Such a rectangle is called the "golden rectangle," or the "golden section;" and this ratio of its width to its length is called the "golden mean."

To express this relationship algebraically, call the length L, and the width kL. Then:

$$\frac{kL}{L} = \frac{L}{L + kL}, \quad \text{and} \quad k = \frac{1}{1+k} = \frac{1}{2}\sqrt{5} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}.$$

Jay Hambidge in his books, "Practical Applications of Dynamic Symmetry," and "The Parthenon and Other Greek Temples, Their Dynamic Symmetry," (Yale University Press), indicates the extent of the utilization of the golden section in Grecian art and architecture. Many articles of trade such as cigar, candy, and match boxes, playing cards, books, tables, cakes of soap, and sugar dominoes are frequently produced in these proportions.

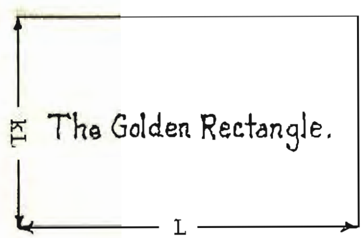
A study of the accompanying sketches will make clear certain important relationships.

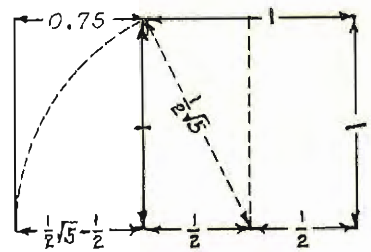
The long side of a golden rectangle is equal to half the short side plus the diagonal of half the square of the short side.

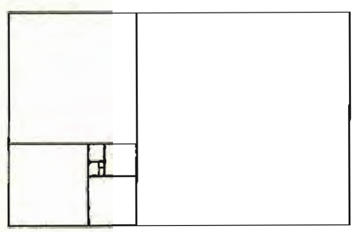
If the square of the short side is cut from one end of a golden rectangle, the remnant is also a golden rectangle. The dimensions of this new rectangle are k times the corresponding dimensions of the original rectangle. If this process is continued, a series of nesting squares is formed in which the sizes of the successive squares are the consecutive powers of k times the original square.

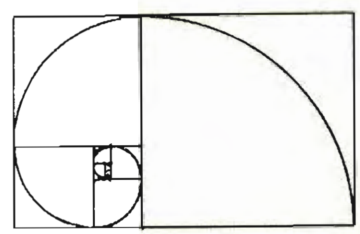
A logarithmic spiral can be inscribed in this figure, as shown. This spiral occurs frequently in nature in the arrangement of leaves on the stem of a plant, the petals of a flower, the pattern of seeds, as in a sunflower head, and in many spiral shells, such as the chambered nautilus.

THE APPLICATION OF THE GOLDEN MEAN









POWERS OF THE GOLDEN MEAN (k)
Duodecimals

Power	Value	Approx.	Power	Value	Approx.
0	1.000 000 000	1.00	0	1.000 000 000	1.00
1	.74Ɛ Ɛ67 728	.75	-1	1.74Ɛ Ɛ67 728	1.75
2	.470 054 494	.47	-2	2.74Ɛ Ɛ67 728	2.75
3	.29Ɛ Ɛ13 254	.2X	-3	4.29Ɛ Ɛ13 254	4.2X
4	.190 141 240	.19	-4	6.X2Ɛ X7X 980	6.X3
5	.10Ɛ 992 014	.11	-5	Ɛ.10Ɛ 992 014	Ɛ.11
6	.080 36Ɛ 228	.08	-6	15.Ɛ3Ɛ 850 994	15.Ɛ4
7	.04Ɛ 622 9X8	.05	-7	25.04Ɛ 622 9X8	25.05
8	.030 948 440	.03	-8	3X.Ɛ8Ɛ 273 780	3X.Ɛ9
9	.01X 896 568	.02	-9	64.01X 896 568	64.02
X	.012 071 X94	.01	-X	X2.ƐX9 Ɛ4X 128	X2.ƐƐ
Ɛ	.008 824 694	.01	-Ɛ	147.008 824 694	147.01
10	.005 449 400	.00	-10	229.Ɛ26 772 800	22X.00
∞	0		- ∞	∞	

The Golden Mean

Among the many ways of expressing the value of the golden mean are the following:

$$k = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\;}}} = \frac{\sqrt{5}-1}{2} = 2 \sin 18^\circ = 0.74Ɛ\ Ɛ67\ 728 = \frac{1}{k+1}$$

The continued fraction is obviously the development of the last of the above expressions. But, by inverting the first and the last expressions in this equation, we discover that k has the peculiarity that the same result is obtained by either adding *1* to k, or dividing *1* by k. That is:

$$1 \div 0.74Ɛ\ Ɛ67\ 728 = 1 + 0.74Ɛ\ Ɛ67\ 728 = 1.74Ɛ\ Ɛ67\ 728 ,$$

and, from the same terms, that $k^2 + k = 1$, whence $k = 1 - k^2$.

The accompanying table of the Duodecimal Powers of k , will make all this quite evident. And, from the table, other facts, even more peculiar, become apparent. It dawns upon us that the successive powers of k form a Fibonacci series. And, from this we learn, that any power of k is the sum of its next two higher powers, or the difference between its next two lower powers.

It may be seen that the various powers of k may be closely approximated in two places of duodecimals. And here we find a remarkable phenomenon. The approximations of the descending powers of k , starting with k^{10}, repeat the exact form of the basic Fibonacci series, but as duodecimal fractions, thus:

.00 .01 .01 .02 .03 .05 .08 .11 .19 .2X .47 etc.

It has long been known that there is an intimate relation between the Fibonacci series and the golden mean, because the successive terms of the series approximate that ratio to each other. That the successive powers of the golden mean generate the basic Fibonacci series becomes apparent for the first time through their statement in duodecimals, and thus stated, the apparent looseness in the initial terms of the series (0, 1, 1, 2) becomes reasonable and valid. To afford a comparison with their statement in decimals, we append that table.

It is evident that the statement of the powers of k in decimal terms does not accommodate a close approximation in two or three decimal places. Nor do the leading terms parallel the basic Fibonacci series.

There is one phenomenon that is common to both of the tables. The odd powers of k have the same fractional ending, whether positive or negative. And the fractional part of the even positive powers is the complement of the fractional part of the negative.

Powers of k , in Decimals			
0	1.000 000	0	1.000 000
1	.618 034	-1	1.618 034
2	.381 966	-2	2.618 034
3	.236 068	-3	4.236 068
4	.145 898	-4	6.854 102
5	.090 170	-5	11.090 170
6	.055 728	-6	17.944 272
7	.034 442	-7	29.034 442
8	.021 286	-8	46.978 714
9	.013 156	-9	76.013 156
10	.008 130	-10	122.991 870

The Fibonacci Series

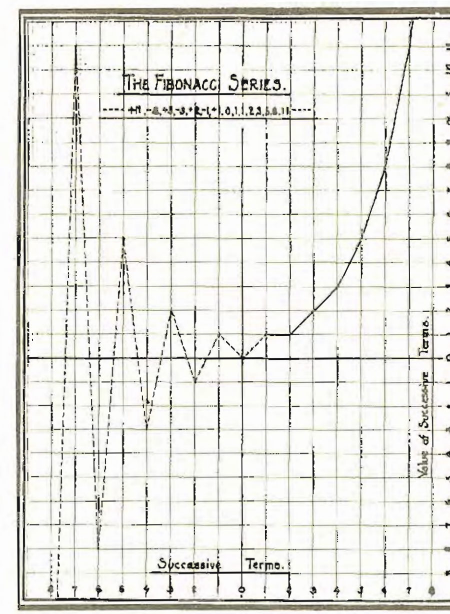
was excellently described and analyzed in Harry C. Robert's article in the February Bulletin. There is no need to go over the same ground again. But this study of the Kin of the Golden Mean lends peculiar emphasis to certain features which should be set forth. However, it could not fail to prove most interesting to review Mr. Robert's article in the light of the present proof that each term of the Fibonacci series equals *100* k^n, and to follow this new thread through the intricate relationships between terms, which he has so well shown.

Eugene P. Northrup points out in his book, "Riddles in Mathematics," (D. Van Nostrand Co.), that, if we arrange the ratios of the successive terms of the series in alternate columns, the values in one column will approach k through values greater than k , while those in the other column approach k through values less than k . In either case, the ratio of the higher terms is almost exactly k = *.74Ɛ Ɛ67 728* .

1:1	= *1.000 000*	*1:2*	= *.600 000*
2:3	= *.800 000*	*3:5*	= *.724 972*
5:8	= *.760 000*	*8:11*	= *.747:474*
11:19	= *.751 86X*	*19:2X*	= *.74Ɛ 364*
2X:47	= *.750 275*	*47:75*	= *.74Ɛ X47*
75:100	= *.750 000*	*100:175*	= *.74Ɛ Ɛ47*
175:275	= *.74Ɛ Ɛ75*	*275:42X*	= *.74Ɛ Ɛ64*
42X:6X3	= *.74Ɛ Ɛ69*	*6X3:Ɛ11*	= *.74Ɛ Ɛ67*
Ɛ11:15Ɛ4	= *.74Ɛ Ɛ68*	*15Ɛ4:2505*	= *.74Ɛ Ɛ68*

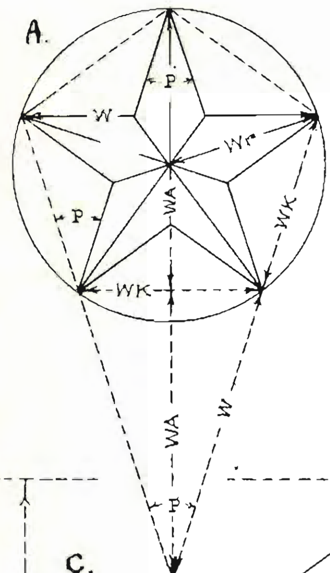
If we use the ratios of each term to its second successive term, in the same way these values will approach k^2, from higher values in the one column, and through lower values in the other. If we use third term ratios, these values will approach k^3. In short, if we use nth term ratios, their value will approach k^n. But in each case these values will alternate between excess and defect, and will approach more and more exact values of k for the higher terms.

The basic statement of the Fibonacci series may also be worded that each term is the difference between the next successive terms. Using this operation, the series may be extended on the low side as follows: *0, +1, -1, +2, -3, +5, -8, +11, -19, +2X, -47* . . The accompanying graph will help to visualize the peculiar oscillation of this extension of the series.

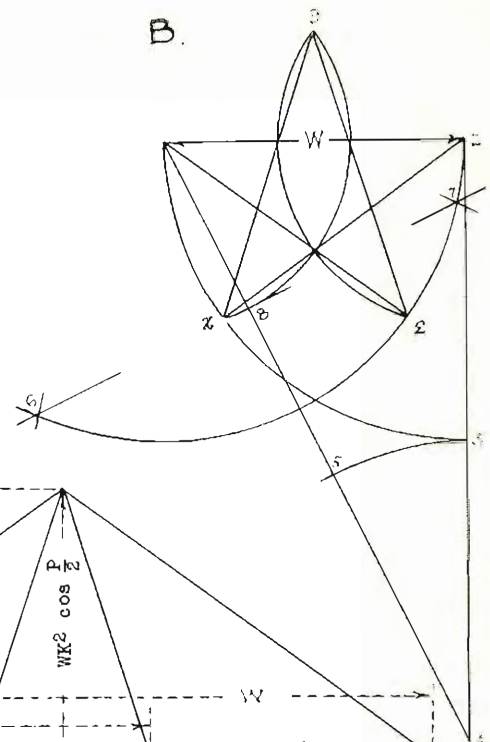


However, since k^∞ equals *0* , this extension of the series seems atypical, and the proper development of the series should lie between *0* and *1* , representing the values $k^{\infty-1}$, $k^{\infty-2}$, etc. Since the logarithmic spiral of the powers of k seems to express the law of growth of living things, perhaps these minutely small values represent the original cells of living organisms, the alternative excess and defect, positive and negative, representing the male and female elements which unite to form the original cell.

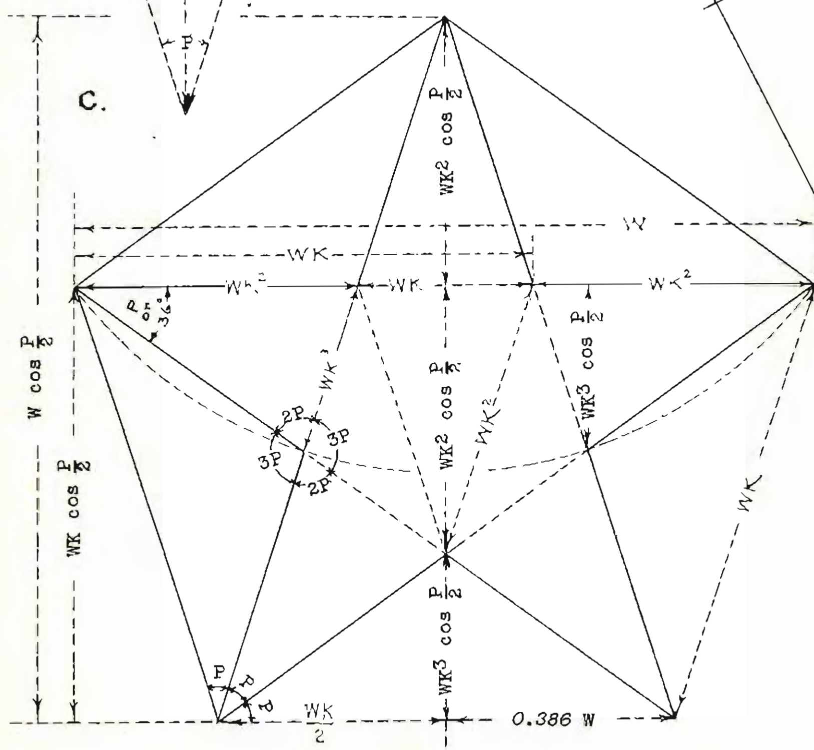
A.



B.



C.



The Pentagon and The Pentacle

One of the expressions for the value of the golden mean is 2 sin 18°, or *2* sin *.07249*ᶜ. Since twice the sin of 18° equals the chord of an angle of 36°, and because the angle (P) of 36° is largely involved in the construction of the pentagon, the pentacle, and related figures, it will be interesting to analyze these constructions.

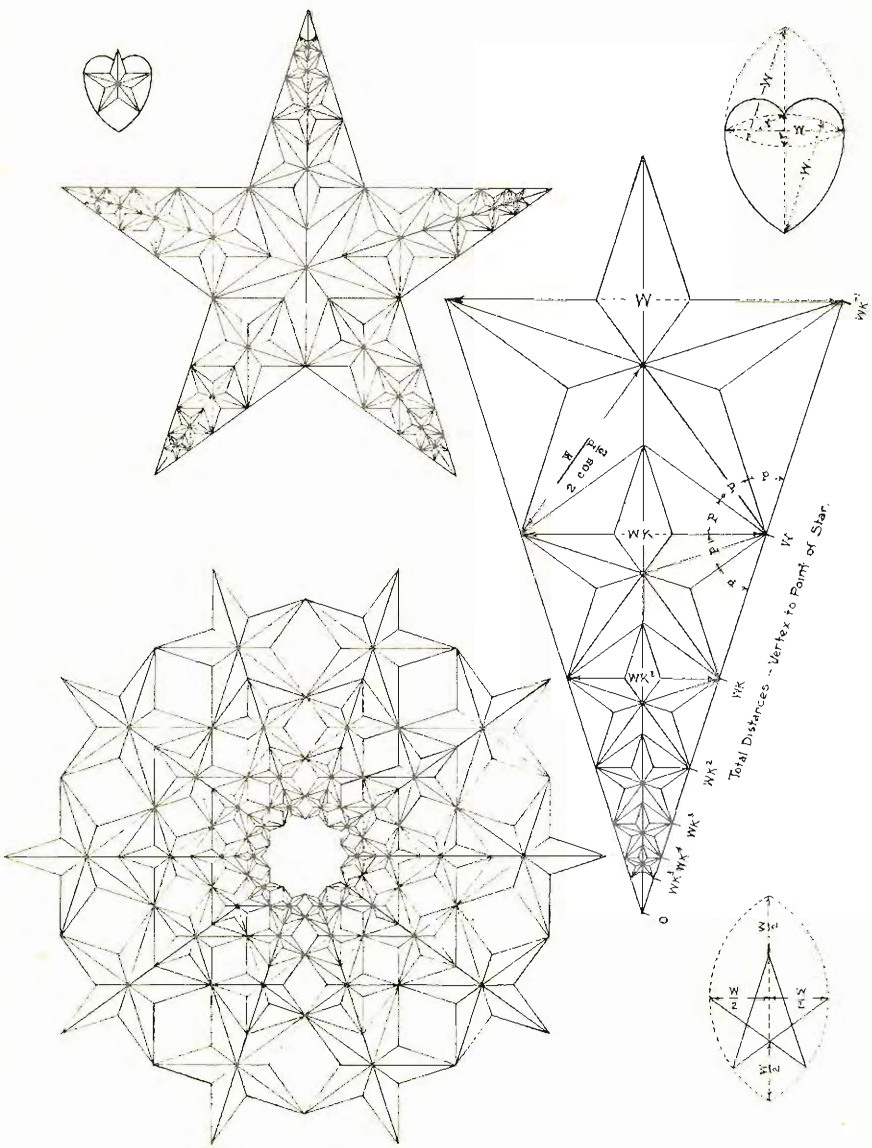
In Figure B we have constructed a pentagon of a given width, W, with ruler and compass. We have numbered the points in the sequence of their determination, so that the steps of construction would be easy to follow.

Using W as the short side, we construct a right triangle whose other leg is 2W, and whose hypotenuse is $W\sqrt{5}$. We subtract W from the length of the hypotenuse and bisect the remainder, which gives us $W/2(\sqrt{5} - 1)$, or kW. This is the length of a side of a pentagon whose width is W. Using kW as a radius, we lay off the other points of the pentagon, and complete the figure. By drawing in the diagonals, we will have inscribed a pentacle within the circumscribed pentagon, and, within the pentacle, another inscribed pentagon.

All of the angles of these pentate figures are multiples of the angle P, which equals one tenth of a circle. Pythagoras was much interested in that isosceles trangle whose base angles were twice the angle at the apex. These angles are relatively 2 P and P. And this triangle is the one developed by extending the sides of the pentagon to form the stellated pentagon. This is variously called the pentacle, the pentagram, or the pentalpha. Many mystical properties have been ascribed to this figure, but we are primarily interested in its geometric characteristics.

An examination of the drawings will show the surprising degree in which the powers of k enter into the elements of these figures, and will emphasize the Fibonacci characteristic that each power of k is the sum of its higher powers; thus:

$$\begin{aligned} k^0 &= 1k^0 + 0k^1 \\ &= 1k^1 + 1k^2 \\ &= 2k^2 + 1k^3 \\ &= 3k^3 + 2k^4 \\ &= 5k^4 + 3k^5 \\ &= 8k^5 + 5k^6 \end{aligned}$$



Even the projected dimensions repeat the Fibonacci characteristics, and in the ratios they bear to their originals there is further reiteration. For instance

$$\sin \frac{P}{2} = \frac{k}{2} = .3860 \quad , \quad \text{and} \quad \cos \frac{P}{2} = \frac{\sqrt{k+3}}{2} = .Ɛ4Ɛ5$$

The elements of the inscribed pentagon are k^2 times those of the circumscribed pentagon. The pentacle drawn within it would be similarly related to the larger pentacle. The elaborations of these ratios possess a singular beauty.

Our study has found some interesting aptitudes of the duodecimal base, and an intricate series of relationships based upon the golden mean. We will restate our findings as a sort of summary.

> Duodecimals afford a convenient abbreviation for the golden mean. Approximately, $k = .75$.
>
> Through the use of duodecimals, and of this approximation, we find that the powers of the golden mean not only constitute a Fibonacci series, but actually generate that series in its basic form.
>
> There is a fundamental relationship between the golden mean, the pentagon, and the pentacle. The elements of these figures, including the angular functions, are functions of the powers of k. (Note that $5.1\,k$ is a good approximation for π .)
>
> Involutions of star designs therefore possess an innate dynamic symmetry, excellently illustrating the application of the golden mean.

Reviewing this genealogical integrity, our pipe-smoking philosopher wonders what the further extension of this kinship might be.

The dodecahedron is the regular solid with twelve pentagonal faces, and there is a companion figure, corresponding to the pentacle, in the stellated dodecahedron.

When the stars are right, we will voyage in that region in the hope of further discoveries.

DUODECIMAL J. G'S.

Member and Mrs. Jamison Handy, Jr., announce the arrival of a daughter, Bettina Marie, whose duodecimal birthday is *1163.6.23*.

Members Robert and Mary Lloyd introduced a second son, Ralph Howard, on *1163.7.4.* His older brother, Robert Crawford Lloyd, Jr., had his birthday *1162.6.15.*

Our congratulations and best wishes to these parents of a new generation of dodekaphiles.

We believe that the Lloyd children may constitute the first members of the *3*rd Generation Dodekaphiles, although there is a wide area of uncertainty. We know that a son of Sir Isaac Pitman was a dodekaphile, but know nothing further. The Terrys and the Seelys may have something to say about this.

It would be a pleasant custom to enroll all children of members into a Junior Grade. They could be matriculated to full membership under the stated conditions. What could be more fitting than the presentation of a duodecimal watch, suitably engraved, upon their matriculation?

FINDING CUBE ROOTS

by George S. Terry

We usually find the arithmetical method of extracting cube roots so laborious that we do not bother to remember it. There is an easy method of finding cube roots by the use of a short table, similar to the method of finding square roots published in the last issue.

Using the appropriate column, take an approximate value from the table by direct proportion.

Divide n by the square of this value.

Adjust the approximate value toward this result by *1/3* the difference between them.

Example: Find $\sqrt[3]{31}$ to six figures, the last being approximate.

From the table, the approximate value for *31* would be the value for *3* plus one-twelfth the difference between the values for *3* and *4*, or *3.37 + .04 = 3.3Ɛ.*

We divide *31* by $3.3Ɛ2^2$.

31 ÷ Ɛ.094 1 = 3.416 36

n	$\sqrt[3]{n}$		
2	131	2X7	673
3	154	337	769
4	171	377	83X
5	186	3Ɛ0	8Ɛ7
6	19X	41Ɛ	964
7	1XƐ	447	X04
8	200	46Ɛ	X5X
9	210	492	XXX
X	21X	4Ɛ2	Ɛ36
Ɛ	228	511	Ɛ7Ɛ
10	236	52Ɛ	1000

One third the difference between *3.32* and *3.416 36* = *.00X 12*

3.3Ɛ + .00X 12 = 3.3ƐX 12

If greater accuracy is desired, divide *31* by the square of *3.3ƐX 12,* and adjust again.

31 ÷ Ɛ.12Ɛ 397 754 4 = 3.3ƐX 063 498

Since this is smaller than *3.3ƐX 12*, one third of the difference is subtracted from that figure.

3.3ƐX 12 − .000 026 X49 = 3.3ƐX 0Ɛ5 173

If n begins with *1*, multiply by *23*, find the cube root of that figure, and divide the result by *3*.

For practice in the method, and to learn what accuracy may be expected, let us try it on known cubes.

Find $\sqrt[3]{X5}$. From the table, *4.Ɛ2 + .5(.1Ɛ)* = *4.ƐƐ7*

$X5 \div (4.ƐƐ7^2 = 2.0Ɛ7\ 021)$ = *5.00X 01*

$4.ƐƐ7 + (\frac{.013\ 01}{3} = .005\ 00)$ = *5.000 00*

Find $\sqrt[3]{160}$. Multiply by *23*, and find $\sqrt[3]{3460}$.

From the table, *15.4 + .46(1.9)* = *15.ƐX6*

$3460 \div (15.ƐX6^2 = 22Ɛ.760\ 130)$ = *16.030 05*

$15.ƐX6 + (\frac{.046\ 05}{3} = .016\ 02)$ = *16.000 02*

This, divided by *3* = *6.000 01*

In the last Bulletin, (Vol. *3*, No. *2*), the table of squares, in connection with the article on "FINDING SQUARE ROOTS," was incorrectly printed.

The corrected table is reprinted herewith, so that you can insert into the previous table the figures for *Ɛ* and *10* which were omitted or incorrectly shown.

n	$\sqrt{n}$	
3	189	600
4	200	6Ɛ2
5	22X	78Ɛ
6	255	85X
7	279	920
8	29Ɛ	997
9	300	X48
X	31Ɛ	XƐ5
Ɛ	33X	Ɛ5X
10	357	1000

COMPENSATING ERRORS

by Harry C. Robert, Jr.

One of the aftermaths of World War II, probably just a part of the general relaxation of nervous tension, is the considerable decrease in what some of us refer to as the "passion for accuracy". Apparently many of us would like to substitute a non-chalant and abiding faith in some mystical law of "compensating errors" for the old fashioned and admittedly more tiresome methods of achieving accuracy. Certainly we should not reject such a magnificent step toward an easier and perhaps better way of life without carefully analyzing the matter. Fortunately that much neglected branch of mathematics, Probabilities, provides very sound but elementary methods for investigating such problems.

Let us assume that we can make only two kinds of errors, A equal to (e) and B equal to (-e), and that in any given instance we are equally likely to make either error. Now for any operation if we make no errors, the total error and the average error will both be zero. But if we make one error, it must be either A or B and, disregarding signs, the magnitude of our total and average errors in either case is (e). Since we make only one error, there can be no "compensating error".

If we now take an operation where we make two errors, we find that there are four possibilities:

1. Both errors are A, or 2A = 2e
2. Both errors are B, or 2B = -2e
3. First error is A, second is B, or A + B = 0
4. First error is B, second is A, or B + A = 0

that is, in two cases the magnitude of our total error is (2e), while in the other two it is zero, or we can say that the probability of "compensating errors" is 1/2 and the probability of a double error, (2e), is the same. Adding the four cases (without regard to signs) and dividing by four, we find that we have an average error of (e).

For an operation involving three errors, we will have eight cases obtained by adding first an A to each of the above four cases and then adding a B to the same original four cases. In like manner we may continue to operations where we make four, five or more errors. A table of the probabilities follows:

NUMBER of ERRORS	PROBABILITY TOTAL ERROR WILL BE (PLUS OR MINUS)							AVERAGE ERROR
	0	e	2e	3e	4e	5e	6e	
0	1/1							0
1		1/1						1.0(e)
2	1/2		1/2					1.0(e)
3		3/4		1/4				1.6(e)
4	3/8		4/8		1/8			1.6(e)
5		X/14		5/14		1/14		1.X6(e)
6	X/28		13/28		6/28		1/28	1.X6(e)
7		2Ɛ/54		19/54		7/54		2.23(e)

Inspection of the table will show that in the zero column, which gives the probabilities for completely compensating errors, (that is, cases in which the total error for the operation is reduced to zero) the probability fractions are decreasing rapidly, the odds being less than one to three that our errors will completely compensate each other if we make as many as six errors in an operation. If we extended the table further we would find that for dek errors the fraction is less than 1/4; for do-four errors, less than 1/5; and for two-do errors it will be less than 1/6. We also see that the average error for an operation increases as the number of errors increases, reaching a value of more than (2e) for seven errors, and if we extend our table we will find that the average error will be greater than (3e) for any operation having do-three or more errors.

The "sure-shot" player will only be interested in those cases where the probability is 1/1, that is, the case of one error where he is sure to be wrong, and the case of zero errors where he is certain of being right. For those who insist on leaving their hopes for accuracy in the lap of "Lady-Luck", our little study does not offer much encouragement. Of course it appears that one should never be satisfied with an odd number of errors since, by making just one more error you apparently do not increase the average error and you then have some chance, (with an even number of errors) of having a total error of zero. This might be just a "come-on" unless you have your errors under such good control that you only make the simple kind on which this study is based. If you must gamble, better figure the odds.

Regardless of your personal feelings about accuracy, you will be interested in another phase of this study, that is, the manner in which it shows the advantages of the dozenal base for such studies. For example, we have shown the value of the average error for the case of seven errors as 2.23(e). This fraction is complete, not rounded off. If we had used the decimal base, the figure would be 2.1875(e), requiring twice as many digits after the fraction point. If, as is frequently done, we had elected to use point-form fractions in lieu of vulgar fractions in our table, the probability of a total error of (5e) in the case of seven errors which we show as *7/54* would have been written *.139*, but using Base X, the fraction 7/64 would be .109375, again requiring twice as many digits after the fraction point. Since the fractions involved are obviously multiples of reciprocals of powers of two, it may be interesting to investigate these reciprocals.

The following observations can be readily made from a little table of the reciprocals of powers of two.

Base X:
1. $(1/2^n)$ has (n) digits in its complete fraction.
2. If the point is moved (n) places to the right, the reciprocal of 2^n becomes 5^n as may be easily shown.

Base XII:
1. $(1/2^n)$ has (n/2) digits in its complete fraction if (n) is even, or (n+1)/2 digits if (n) is odd.
2. If (n) is even, the reciprocal of 2^n is equal to $3^{n/2}$ and if (n) is odd, it will be $2(3)^{(n+1)/2}$

Comparing (*1*) we see that when (n) is even, the reciprocal of 2^n has only *1/2* as many places in base twelve as it has in base ten. For (n) odd, the number of places dozenally differs from *1/2* of the decimal places only by a fraction. Comparing (*2*) we find that when initial zeros are eliminated, the base ten fractions require from two to three or even more times as many digits as are needed in the dozenal base, since the first increases by the multiplier *5* for each power of *2*, while the second increases by the multiplier *3* for each two powers of *2*. This is of great importance when our fractions are to be involved in some operation such as multiplication.

Since our little study of "compensating errors" is typical of certain types of probability problems, it is obvious that the dozenal base is a far more efficient tool for such problems than we will find in the decimal system. Many other types of probability problems involve factorials rather than powers of two,

and for these problems also there is a great advantage in using the dozenal base. Mr. Terry has already given us a very convincing picture of the greater efficiency of the dozenal base for handling factorials, in the "Dozen System".

MORE ABOUT SQUARE SUMS OF CONSECUTIVE SQUARES

By The Research Committee

As stated in the introductory paper on this subject, the sum of N consecutive squares is $N\left[a^2 + \frac{N^2 - 1}{10}\right]$ where "a" is the root of the mid-square if N is odd or is the mean of the roots of the two mid-squares if N is even. To find those values of the sum which are squares we may let the sum equal b^2 and write -

$$N\left[a^2 + \frac{N^2 - 1}{10}\right] = b^2 \qquad \text{Eq. } (1)$$

an intermediate equation with unknowns: a, b and N. Because of the number of unknowns and particularly since one of them, N, appears as a third power, there is not available any direct means of solving Eq. (*1*). The most feasible approach to a solution is to limit N to certain specific algebraic or numerical values which substituted in Eq. (*1*) produce equations of the type -

$$b^2 - Da^2 = H \qquad \text{Eq. } (2)$$

where "a" and "b" are unknowns, D equals the non-square factors of N and H is a numerical or algebraic quantity dependent upon N but constant for any specific value of N. Since means are available for solving Eq. (*2*) or proving that there is no solution, we may, by solving equations of this second type for various values of N find solutions of Eq. (*1*).

Before discussing the methods of finding solutions of these equations, mention should be made of the extensive preliminary research by George S. Terry which provided the foundation on which all of the developments of this subject are based. This work consisted of considering N in terms of its final digits, first, finding what values of "a" will produce square endings for the left side of Eq. (*1*); second, determining what values of "a" make the quantity within the brackets on the left side of Eq. (*1*) divisible by such non-square factors of N as remain outside the brackets after the quantity inside has been made integral; and third, the finding of initial solutions for those values of N which appear possible. As an example of these operations, we take -

$N = 61$

The Sum $= 61\left[a^2 + 310\right] = b^2$, if square.

By inspection we see that if -

a^2 ends 00; bracket ends 10; sum ends 10.

a^2 ends 30; bracket ends 40; sum ends 40.

a^2 ends, even 1; bracket ends, odd 1; sum ends, odd 1.

a^2 ends 90, 69; bracket ends 19, 79; sum ends 79, 19.

a^2 ends 14, 54, 94; bracket ends 24, 64, $\mathcal{X}4$; sum ends 24, 64, $\mathcal{X}4$.

None of which give square ends for the sum. But if -

a^2 ends 04, 44, 84; bracket ends 14, 54, 94; sum ends 14, 54, 94.

Since this last case provides possible square endings for the sum, $a = (10k \pm 2)$ is possible since only this form will produce an a^2 with endings of 04, 44, 84.

The next step is learn if $\left[a^2 + 310\right]$ is divisible by 61 and the form of "a" for which this is possible. If $N = 61$ is a possible solution of our problem -

$$\text{some integer, } p = \frac{a^2 + 310}{61} = 6 + \frac{a^2 + 6}{61}$$

$$\text{and } \frac{a^2 + 6}{61} = q, \text{ an integer}$$

$$\text{and } a^2 = 61q - 6 = 61q_1 + 57$$

This last equation is solved by adding multiples of 61 to 57 that will produce square endings in the sum until a square (not greater than $(61/2)^2$) is found. Thus -

	57	
add 2 x 61	102	
	159	not square
add 3 x 61	163	
	300	not square
add 61	61	
	361	not square
add 3 x 61	163	
	504	not square
add 5 x 61	265	
	769 = $(\pm 29)^2$	

From which we learn that when $a = 61k_1 \pm 29$, the bracket will be divisible by 61. Since 61 is prime there will be no other form, but if the divisor is composite, the above procedure must be carried out to the square of one-half of the divisor, as in some cases there may be other forms that meet the divisibility requirement. If, for any N, no square is found within the above limit, there is no value of "a" for which the bracket is divisible and that value of N must be eliminated.

For $N = 61$ however the first two requirements have been met so we procede with the third operation, the finding of actual solutions, first combining our two forms for "a". We now know that -

$$a = 10k \pm 2 = 61k_1 \pm 29$$

$$10k = 61k_1 \pm 27$$

$$k = 6k_1 \pm 2 + \frac{k_1 \pm 7}{10}$$

now $\dfrac{k_1 \pm 7}{10} = k_2$, an integer, from which $k_1 = 10k_2 \mp 7$

and $a = 61(10k_2 \mp 7) \pm 29 = 610k_2 \mp 33\mathcal{X}$

Now, by inverting the double sign in one of our forms and using a similar procedure we find that, also -

$$a = 610k_2 \pm 8\mathcal{X}$$

These two forms for "a" with their double signs now give us four series for "a" to be checked for solutions. Let us take $a = (610k_2 + 8\mathcal{X})$ and substitute it in the equation for the sum -

$$\text{Sum} = 61\left[a^2 + 310\right] = 61\left[(610k_2 + 8\mathcal{X}) + 310\right]$$

$$= 61^2\left[6100k_2^2 + 1580k_2 + 114\right]$$

$$61^2 \cdot 4^2\left[469k_2^2 + 113k_2 + \mathcal{X}\right]$$

We may tabulate this last value -

k_2	a	Sum		1st Diff.	2nd Diff. (Constant)
0	8X	$61^2 4^2 \cdot$	X		
				580	
1	69X	"	58X		916
				1296	
2	10XX	"	1864		916
				1ƐƐ0	
3	16ƐX	"	$3854 = (68)^2$		

Thus we find that 61 consecutive squares of which the mid-square is $(16ƐX)^2$ have a sum of $61^2 \cdot 4^2 \cdot 68^2$.

Of particular interest to us is the manner in which the relative simplicity and regularity of square endings in the dozenal base has contributed to reducing the work in each of the three steps of the preliminary investigation. The importance of this feature is well illustrated by the fact that whereas Dickson's "History of the Theory of Numbers" records no investigations of this problem for values of N greater than *131*, Mr. Terry, using the dozenal base, investigated all values of N up to 600. This work not only eliminated most of the impossible values of N but also provided one or more solutions for most of those values of N which met the requirements of the first two steps. Observed patterns among these solutions led to several interesting extensions which apparently have not been reported previously.

The restrictions of N developed by this preliminary work may be summarized -

1. N ending *0* must be of the form $20k^2\ (10p + 1)$
2. N ending *1* must be of the form $(20p + 1)$
3. N ending *2* must be of the form $(20p + 2)$
4. N ending *4* must be of the form $(20p + 14)$
5. N ending *9* must be of the form $(20p + 9)$
6. N ending Ɛ may have either even or odd digit before Ɛ.
7. N, if square, may only be of the form $(6n \pm 1)^2$
8. No other values of N will produce square sums.

By no means do all values of N of the first six forms listed above meet the requirements for producing square endings or divisibility. Although not yet proven, it appears that N must always be of the form $(\pm 1)\ (t^2 - 3u^2)$ which has the effect of eliminating any N having non-square factors of the form $(10k \pm 5)$.

Returning to the question of solving -

$$b^2 - Da^2 = H \qquad \text{Eq. } (2)$$

we find that we have two distinct cases -

1. D = *1*, that is, N is a square, and
2. N has non-square factors.

The first case has a strictly limited number of solutions for each N whereas the second has in every case an infinite number of solutions. Since the two cases are entirely different in the methods of solution they must necessarily be treated separately, so we will first dispose of the case of N, a square, before undertaking the more complicated second case.

SQUARE SUMS OF A SQUARE NUMBER OF CONSECUTIVE SQUARES

If N is an even square, the sum, $N\left[a^2 + \frac{N^2 - 1}{10}\right]$, may be written $\frac{N}{4}\left[(2a)^2 + \frac{N^2 - 1}{3}\right]$ where N/4 is an integral square; (2a) is an odd integer whose square ends *1* or *9*; and $\frac{N^2 - 1}{3}$ is not integral for N ending *0*, or is an odd integer (ending *1*, *5* or *9*) if N ends *4*.

Thus $\left[(2a)^2 + \frac{N^2 - 1}{3}\right]$ is either not integral or is a singly even number (ending *2*, *6*, or X) and cannot be an integral square.

Therefore a square sum is impossible with N an even square.

If N is an odd square, in the sum, $N\left[a^2 + \frac{N^2 - 1}{10}\right]$, "a" is integral; and $\frac{N^2 - 1}{10}$ can only be integral if N ends *1*, that is, if $N = (6n \pm 1)^2$

Thus we prove that if N is a square number it can only be of the form $(6n \pm 1)^2$

Now since N is square we may write $b = c\sqrt{N}$ or $b^2 = Nc^2$, and substituting in Eq. (1) -

$$\text{Sum} \quad N\left[a^2 + \frac{N^2 - 1}{10}\right] = b^2 = Nc^2 \quad \text{which reduces to}$$

$$c^2 - a^2 = (c + a)(c - a) = \frac{N^2 - 1}{10}$$

and the number of solutions will depend on the number of decompositions of $\frac{N^2 - 1}{10}$ into two even factors which are then equated to $(c + a)$ and $(c - a)$ respectively and the resulting simultaneous first degree equations solved for "a". The greatest value of "a" will be found when $(c - a) = 2$, this solution being -

$$a(\text{max.}) \quad \frac{N^2 - 41}{40} = \frac{1}{4}\left[\frac{N^2 - 1}{10}\right] - 1$$

and since it may be shown that $\left[(6n \pm 1)^4 - 41\right]$ is always divisible by 40, we thus prove that $N = (6n \pm 1)^2$ is always possible.

As mentioned, the number of solutions will depend on the factorization of $\frac{N^2 - 1}{10}$ for if -

$$\frac{N^2 - 1}{10} = 2^d p^e q^f r^g \ . \ . \ .$$

the number of solutions will be -

$$Z = \frac{(d - 1)(e + 1)(f + 1)(g + 1)}{2} \ . \ .$$

of which we eliminate those values of -

$$a < \frac{N + 1}{2}$$

if we wish to include only those solutions where the smallest of the N squares is greater than 0.

For example, $N = (27)^2 = 681$

$$\text{Sum} = 681\left[a^2 + \frac{(681)^2 - 1}{10}\right] = b^2 = 681c^2$$

or $c^2 - a^2 = 38654 = 2^5 \cdot 5 \cdot 11 \cdot 31$

which gives a total of 14 solutions of which 6 give values less than $\frac{N + 1}{2}$, and the greatest solution is -

$$a(\text{max.}) = \frac{38654}{4} - 1 = Ɛ174 - 1 = Ɛ173$$

A list of solutions for $N = (6n \pm 1)^2 < 600$ and $a \geq \frac{N + 1}{2}$ follows:

N	a	N	a	N	a
N = 41	a = 41	N = 201	a = 118	N = 381	a = 373
			1Ɛ8		3Ɛ9
N = X1	a = 214		247		809
			2ƐƐ		1829
N = 121	a = 96		401		3459
	416		604		
			100Ɛ	N = 441	a = 431
					242X
		N = 261	a = 38X		4861
			632		
			16X2	N = 5X1	a = 126X
					1852
					863X

THE ADVANTAGES OF DUODECIMALS

by George S. Terry

As we explore the dozen system of counting, many advantages appear. In all comparisons with the ten system, it is important to distinguish between those gains which are fundamental, and those which are fortuitous.

Few will question the fundamental advantages of the duodecimal multiplication table, or the duodecimal division of the unit circle. But when we come to abbreviations of incommensurate numbers, the actual error, due to rounding off, may be greater or less duodecimally, depending on the figure picked up or omitted, and any improvement in accuracy is purely fortuitous.

The example given for the abbreviation of π in Vol. 3, No. 1, of the Bulletin is a very fair one, as may be seen from the extended values there given. However, the advantages must be regarded as fortuitous, for, if we stop at the second place, the 3.18 duodecimal is not a better approximation than the 3.14 decimal.

The fundamental advantage in any rounding off, is shown by the maximum error which can be introduced at corresponding places, and, in general, if we round off at the whole number, the maximum error is one half. If we round off after the first place, the possible error is one in a score, as compared with one in two dozen; if after the second place, it is one in two hundred, as compared with one in two gross; and so on, with progressive advantage in the duodecimals. From the fourth place on, the possible error can be only less than half duodecimally, what it can be decimally. With about a dozen places, we get greater accuracy in duodecimals with one less place than the decimal. This is a fundamental advantage.

The answer to the Fibonacci problem quoted in the same number of the Bulletin is another interesting example, but this, too, must be regarded as not fundamental. Three squares in arithmetic progression with a common difference of 5, are:

duodecimally	$(2.7)^2$	$(3.5)^2$	$(4.1)^2$
or, decimally	$(2.58\dot{3})^2$	$(3.41\dot{6})^2$	$(4.08\dot{3})^2$

Kraitchik indicates that there is an infinite number of solutions, the second being, duodecimally:

$$\left(\frac{5567Ɛ}{600ƐX0}\right)^2 \qquad \left(\frac{1153341}{600ƐX0}\right)^2 \qquad \left(\frac{1700141}{600ƐX0}\right)^2$$

Suppose the problem had been to find three squares with a common difference of 6. We would have:

duodecimally	$(.6)^2$	$(2.6)^2$	$(3.6)^2$
or, decimally	$(.5)^2$	$(2.5)^2$	$(3.5)^2$

Or, for a common difference of 7, the figures would be:

duodecimally	$(.Ɛ\dot{3}724\dot{9})^2$	$(2.98\dot{4}97\dot{2})^2$	$(3.X\dot{3}724\dot{9})^2$
or, decimally	$(.941\dot{6})^2$	$(2.808\dot{3})^2$	$(3.841\dot{6})^2$

THE MAIL BAG

Anent our request for a more suitable term for what we call duodecimal-form fractions, Lewis Carl Seelbach first preferred the term "Partials," and then, a little later, suggested the word "Basicules." A friend who prefers to remain anonymous, has suggested "Fractionals." We will appreciate receiving any other suggestions that you may have.

During the vacation season, we received quite a few requests to mail duodecimal literature to acquaintances that our members had interested while on their trips. This leads us to suggest that copies of The Dozen System, and of New Numbers, should make excellent Christmas gifts. We can assure you that they will be mailed promptly and properly to any names you may give us.

Kingsland Camp called our attention to a quotation in the Reader's Digest for October that he feels has special meaning for us:-

> "The reasonable man adapts himself to the world; the unreasonable one persits in trying to adapt the world to himself. Therefore all progress depends upon the unreasonable man.
>
> George Bernard Shaw."

This is especially apt since G.B.S. has made several favorable comments on the duodecimal base in his writings. We wish that he were as aware of us as we are of him.

The recent appearance of the United Nations Flag reminds us to remark that flags for the United Nations have been proposed by Lewis Carl Seelbach, and by Col. Robert S. Beard, who makes his first duodecimal appearance in this issue. We should also note that Alfred Norland has submitted to that organization a proposal for an international currency.

Informally, we keep in touch with the activities of the American Standards Association and of the United Nations Standards Co-ordinating Committee. This Fall has seen the formation of the International Organization for Standardization (ISO), and the establishment of co-ordination between it and the UNESCO. All of the work done on international projects by the ISO, and acceptance of the resulting recommendations in each member country, will be voluntary. It is probable that the ISO will in large measure supersede the UNSCC. The ISO replaces the wartime International Federation of National Standardizing Associations (ISA) which established as a world standard the 25.4 inch-millimeter conversion ratio.

In December 1946, the International Civil Aviation Organization was formed and early showed a preference for metric standards. It is now announced that it will co-ordinate closely with the ISO and will leave to that body the establishment of material and manufacturing standards. God's in his heaven, and the world's not as bad as it might be.

We want to share with you the pleasure we got from a letter of John Selfridge's. Here it is:-

"Dear Mr. Beard,

"Iam writing this from a pay typewriter at the University of Washington Library. The librarians just got around to cataloging the *Bulletin*, or at least it was not in the Catalog a few weeks ago when I checked out (for the second or third time) Mr. Terry's *Arithmetic*. I discovered the *Bulletin* yesterday, and spent the afternoon and evening devouring the first seven issues (to June '47). I have not been so enthusiastic about anything since coming to the University two years ago.

"I have had an interest in duodecimals (or I should say base 12) since I first asked myself what our number system was based on, but it was brought to a head by reading Mr. Terry's *Arithmetic*, and Mr. Norland's *The Twe System*. The thing that strikes me as funny about it all is that almost everyone who tries to develop the *best* system of positional notation decides on base XII but from there on in it is hard to be completely scientific about it.

"Right now I am going to reverse the decision I made before writing you and give you a load of my opinions even tho I am not yet a member.

"As far as the system of least change is concerned, I cannot escape the conclusion that it is somewhat disappointing to discover, after being told there are just two new symbols, that you must go back and learn your arithmetic *all* over again, at least as concerns notation. Isn't it true that the whole world must be reeducated if the system is going to do us any good? I maintain (contrary to Mr. Norland and others) that our job is indeed quite comparable to that of the protagonists of Esperanto or Basic English and even greater due to the fact that advocates of base XII will admit that eventually notation base X will be dropped completely from everyday life and learned only by perusers of Antique books, etc.

"The first and most obvious point of difference of those using the system of least change is the choice of two new symbols. In the final analysis, they are almost completely arbitrary. I will make no attempt to rank the conditions of choice in order of importance but will list just a few of them. On second thot I won't, as that sort of thing would extend my letter indefinitely. (You see I am human after all.)

"Next letter I will try to give my analysis of why I agree with those who would like to use small delta and epsilon, which in informal typewriter correspondence could be slurred into d and e, making everything all lower-case-and-real-nice. (There are a number of good healthy objections, tho.)

"Well, there goes my last dime into the slot, so I'll have to finish this letter within the half hour. Let me say this right now: I feel that I have no work of more importance in my life ahead than becoming one of the most active members of the Society and furthering the idea of base XII notation. I plan to get a B.S. in Math. next year--I am a member of the Math. honorary--I plan eventually to do research in Number Theory--all these seem rather insignificant or temporary alongside the fact that I might become one of the first gro members of the Society and help to build it up to its proper dimensions.

"My main idea is this: we can find no better proponents of base XII than those who learn it as their *first* arithmetic and then are forced to learn the clumsy base X system. They don't want a system of least change, what they want to start out with is the system perfected, with no traditions or customs governing the choice of nomenclature. More about this next time, as my typing time is running short.

"I am very interested in obtaining the first eight issues of the *Bulletin*, as I would like to have a complete file of them. Please send them and bill me, or tell me how much to send in advance.

"I am almost certain that there are good potentialities of a duodecimal group here at the U. I haven't done much inquiring but I'm sure I could get it organized or at least a few more really interested new members for the Society.

"Wish I could take time to write a few more pages, but am equally anxious to get this in the mail.

"Hoping that I may soon be admitted as a member, I am very truly yours,

John Selfridge."

Naturally, we started things moving for Mr. Selfridge right away. But what we wish to remark is that, if some one would supply us with a stimulant like this from time to time, there would be a noticeable improvement in the job done by

Ye Ed.