The Duodecimal Bulletin

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THE DUODECIMAL SOCIETY OF AMERICA

20 Carlton Place ~ ~ ~ ~ ~ Staten Island 4, N. Y.

THE DUODECIMAL SOCIETY OF AMERICA

is a voluntary nonprofit organization for the conduct of research and education of the public in the use of Base Twelve in numeration, mathematics, weights and measures, and other branches of pure and applied science.

Full membership with voting privileges requires the passing of elementary tests in the performance of twelve-base arithmetic. The lessons and examinations are free to those whose entrance applications are accepted. Remittance of $6, covering initiation fee ($3) and one year's dues ($3), must accompany applications.

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The Duodecimal Bulletin

SIR ISAAC PITMAN ON THE DOZEN SYSTEM

A RECKONING REFORM

From the "Bedfordshire Independent," 24 November

I have been highly gratified with the letters on education, and on a phonetic (that is a strictly alphabetic) orthography, that have for many weeks appeared in your columns; and now another boon is presented by "Vox Clamantis," in Duodecimal Arithmetic, as a substitute for Decimal Money, *which Lord Overstone, and the Queen's Commission on a decimal currency, and the London bankers cannot work.* The introduction of decimal money to supersede our present duodecimal money, (as to the penny and shilling, which are the centre and foundation of our coinage,) would be an act unbefitting any nation acquainted with the elements of mathematics and the properties of numbers. It would be a needless and most hurtful alteration; we should be giving up a good number for a bad one. Until we can get a quarter of *ten* in whole numbers, decimal money, weights and measures must be voted an absurdity.

In the number TWELVE, with the Arabic notation, and its marvelously simple and effective expedient of marking the place of vacancy by a "0" when returning upon the number assumed as the basis of calculation, adding to the ten ordinary figures "T" modified to "↊" for *ten*, and "E" altered to "↋" for *eleven*, (with which I am able to supply your compositor,) we have a universal arithmetic, adapted to our present money, (except the £1, which *must* be given up as to its name, and might be as to its value,) adapted to our present measures, the inch and foot, adapted to our time in some respects, and which may be adapted to our weights by merely dividing the avoirdupois pound into *twelve* instead of *sixteen* ounces.

The nomenclature and the practice of duodecimal counting, as to its general principles, are already in existence in the "dozen" and "gross,"—numbers which on account of their convenience, arising from their divisibility, are more used in trade for counting articles than are "tens" and "hundreds," although our arithmetic is based upon ten. To the universally recognized names "ones" (or units), "dozens," and "grosses," to express the values of the first three figures in arithmetic, we have only to add "triples" or the third power, and "miliads" (a variation of "millions,") for the fourth and seventh figures, (or any other more appropriate names,) and our numeration table is complete; thus,

miliads	grosses of triples	dozens of triples	triples	grosses	dozens	ones
8,	4	3	7,	9	6	2

10 represents one dozen; 11, one dozen and one; 28, two dozen and eight; 347, three gross four dozen and seven; 4,61Ɛ, four triples, six gross, one dozen and eleven; 7,654,321, seven miliads, six gross, five dozen and four triples, three gross, two dozen and one. A triple (1,000) is 1,728 of the present notation. A convenient nomenclature between *twelve* and a *gross*, might be formed by linking the dozens and units by means of the final syllable of "d*ozen*;" thus, as for six tens and eight we say *sixty-eight* (*ty* being a contraction of *ten*) so for four dozen and three, written duodecimally 43, we might say *fourzen-three*; 57 *fivezen-seven*, etc. *Seven* and *eleven*, the only words in counting a dozen that are not monosyllables, might be, in these compounds, contracted to *sen* and *lev*; as we say *senit* (se'nnight) for *seven-night;* thus, 75 would be *senzen-five*; Ɛ9, *levzen-nine*.

Let all numbers be written down in this manner, keeping our twelve pence in the shilling, twelve inches in the foot, and divide the pound into twelve ounces, and *all ciphering becomes child's play with plain figures*, and is performed by the simple rules of Addition, Subtraction, Multiplication, and Division. There are no more divisions of pounds, shillings and pence, with 12 and 20 as divisors; of ounces, pounds, quarters and hundreds, with 16, 28 and 4 as divisors; of inches, feet, yards, rods and furlongs, with 12, 3, 5½, 40 as divisors; and numerous variations of these weights and measures, as applied to different articles in different parts of the country; but instead of this Babel of figures, the most complicated arithmetical operation, such as a calculation of the measurements of a house, and cost of building, with masons', carpenters', painters', and other work, will be performed by the simple rules of arithmetic.

The following are the tables of money, weights and measures which such a revolution in arithmetic would require.

MONEY

4 farthings	make	1 penny
10 (twelve) pence	. . . . .	1 shilling
10 (twelve) shillings	. . .	1 mark
10 (twelve) marks	. . . . .	1 banco

The banco means a bank-note of the value of twelve marks. or £7 4s., and might take the place of our present £5 note, 38,614½p.=3 dozen and 8 bancos, 6 marks, 1 shilling, and 4½d.

WEIGHTS

10 (twelve) ounces	make	1 pound
1,000 (triple) pounds	. . . .	1 load

No denomination is wanted between the pound and the load, except dozens of pounds, and grosses of pounds, or, in the curt language of the market, "dozens" and "grosses."

LONG MEASURE

10 (twelve) inches	make	1 foot
1,000 (triple) feet	. . . . .	1 walk
1,000,000 (miliad) feet	. . . . .	1 distant

The "walk," nearly one-third of a mile, might be employed to measure geographical distances, and the "distant" to measure astronomical distances. No denomination between a foot and a walk, (except the yard of 3 feet,) and between the "walk" and the "distant" would be necessary. Intermediate distances, such as 76Ƨ feet, would be called 7 gross, 6 dozen,aand Ƨ feet. The circumference of the earth, reckoned at 25,000 miles, would be duodecimally, 38,000 walks, nearly as comprehensible a number. The distance of the earth from the sun, reckoned at 95,000,000 miles, would be, duodecimally, 81,000 distants—a much more comprehensible number, keeping in mind that a distant is about 560 miles.

LIQUID MEASURE

10 (twelve) drams	make	1 pint
1,000 (triple) pints	. . . .	1 cask

TIME

10 (twelve) seconds	make	1 prime
10 (twelve) primes	. . . .	1 round
10 (twelve) rounds	. . . .	1 beat
10 (twelve) beats	(of 2 hours each)	1 day

A "prime" is five-sixths of the present minute.

I will now apply duodecimal arithmetic to the business of life, in one or two instances. Until the duodecimal multiplication table is learned, the common decimal multiplication table may be employed, the products being turned into shillings and pence by the pence table.

What is the value of 347 (3 gross, 4 dozen, and 7) articles at 9s. 4d. each?

New Method

347
94
1164
2653
27,694

Answer.—2 dozen and 7 bancos, 6 marks, 9 shillings, and 4 pence. Contrast this with the present method of working. 3 gross, 4 dozen and 7 are four hundred and eighty-seven.

What is the value of 1761 (1 triple, 7 gross, 6 dozen and 1) yards of cloth, at 16s. $7\frac{1}{2}$d. per yard. 16s. $7\frac{1}{2}$d. are 1 mark, 4 shillings, and $7\frac{1}{2}$d.

New Method

1761
$147\frac{1}{2}$
↊467
6604
1761
$990\frac{1}{2}$
$230,377\frac{1}{2}$ with 34 figures.

Or 2 gross and 3 dozen bancos, 3 marks, 7 shillings and $7\frac{1}{2}$d.

Old Method

487
9
4383
4d. = $\frac{1}{3}$ 162 4
20) 4545 4
£227 5s. 4d.

The last operation requires 27 figures, and the other 10, In addition to this, in the old method several mental processes must be effected, while the other is a plain multiplication sum.

Old Method

1761 in duodecimals are 2809 in decimals.

2809
16
16854
2809
6d. = $\frac{1}{2}$ 1404 6
1d. = $\frac{1}{6}$ 234 1
$\frac{1}{2}$d. = $\frac{1}{2}$ 117 0 $\frac{1}{2}$
20) 46699 $7\frac{1}{2}$
£2334 19s. $7\frac{1}{2}$d. with 59 figures.

Or 2809
10s. = $\frac{1}{2}$ 1404 10
5s. = $\frac{1}{2}$ 702 5
1s. = $\frac{1}{5}$ 140 9
6d. = $\frac{1}{2}$ 70 4 6
1d. = $\frac{1}{6}$ 11 14 1
$\frac{1}{2}$d. = $\frac{1}{2}$ 5 17 $0\frac{1}{2}$
£2334 19s. $7\frac{1}{2}$d. with 62 figures.

The same contrast of length and complexity on the one hand, with brevity and simplicity on the other, would be found in calculating lengths, liquids, etc. If anyone should say that the introduction of duodecimal arithmetic, although it would be manifestly productive of so many benefits, is a practical impossibility, I would only reply, "This is the year 1857; see what has been effected, and then doubt of the practicability of anything that is good if you can." A desire to simplify our money and accounts is now extensively felt. That something must be done, everybody who understands arithmetic is convinced. The writer of the article "Weights and Measures," in the "Penny Cyclopaedia" says, "The subject of weights and measures is one the actual state of which is prosperous in the inverse ratio of the number of books or the length of articles which are written upon it. There is nothing in it which might not, if the most natural and simple system were adopted, be described in a very few pages. We are speaking of course only with reference to a possible time; for, let that time arrive when it may, the history of the past must be a confused and repulsive subject." The present may be the beginning of the "possible time" if we like to make it so. The work must commence with the young, and with those few adults who look forward and out of themselves, and not backward and to themselves alone. Duodecimal arithmetic may be introduced into all schools as a mental exercise, just as calculations in several scales are practised in our universities; and when a conviction of the superiority of twelve as a basis has become general, it will be employed in business accounts by those men who when youths at school acquired facility in working duodecimal arithmetic.

Bath, England, 14 November, 1857 ISAAC PITMAN.

DONATIONS AND CONTRIBUTIONS

We wish to express the warmth of our gratitude to Herbert K. Humphrey for his gift of \$50.00 to the Society, and to George S. Terry for a further contribution of \$500.00. There is only one real acknowledgment for these generous donations, and that is our work for the growth and progress of the Society.

The contribution of ideas, of papers and articles for publication, the time spent in looking up references, our talks to the small groups of interested people in our communities, the letters written to the newspapers, and to prominent people to inform them about duodecimals, . . . these are other forms our giving can take. The Society grows by the work of all of us.

DISINGENUOUS DISSUASIONS

by Ralph H. Beard

It is regrettably true that able and learned persons upon occasions cloak the speciousness of their reasoning in profound language in order to conceal the weakness of their position. These statements sometimes become, by frequent repetition, accepted as truths, and serve to confuse man's understanding. Whenever possible, such subterfuges should be unclothed, and the meager skeletons of their structure laid bare.

Several such insidious utterances are attributable to members of the commission of the Academie Francaise which created the French Decimal Metric System. This was in the period of the French Revolution, and it is understandable that these men should have been concerned that their statements should not create resentment in the minds of the populace. Eric Temple Bell reports of this commission that Laplace, Lavoisier, and others, were "purged" out of their seats, but that Lagrange remained as president - "unaware that his gift of silence had saved not only his seat, but also his head."

These were illustrious minds, and one part of their work stands as a milestone in man's progress. They recognized the advantages in establishing the scale of numeration and the scales of the new weights and measures upon the same base. Unfortunately, they determined to base the weights and measures upon the decimal scale of the number system, instead of basing the scale of notation upon the duodecimal base commonly used in their weights and other denominate measures.

The selection of the decimal base was made despite their recognition of the advantages of the duodecimal. Several members of the commission, in defending this action, advanced reasons which add no lustre to their names.

Pierre Simon Laplace, the famous astronomer, said in his Exposition du Systeme du Monde, "Our arithmetical scale is not divisible by three and four, two divisions whose simplicity renders them very common. The addition of two new characters is sufficient to procure this advantage, but a change so important would have been infallibly rejected with the system of measures which was attached to it. Moreover, the duodecimal scale has the inconvenience of requiring that *we retain the products of twelve numbers, which surpasses the ordinary length of memory to which the decimal scale is proportionate.*"

One begins to smile upon reading the statement we have italicized, and the more one ponders its import, the broader becomes

the smile. Today's school-children would be delighted by this recognition of their advanced ability.

Joseph Louis Lagrange, one of the world's greatest mathematicians, advanced the argument that eleven would make a better base than twelve, and would have the advantage that it would give all fractions in the system the same denominator: - that, with any prime base, every systematic fraction would be irreducible and would therefore represent the number in a unique way.

Because of Lagrange's unquestioned ability, this statement deserves most careful analysis. First let us make clear what type of fraction is meant. Regardless of the number base that is used, one can divide anything into as many parts as may be desired, and can represent such parts as fractions of the type:

$$\frac{2}{3}, \quad \text{or} \quad \frac{1}{2}, \quad \text{or} \quad \frac{3}{5}, \quad \text{or} \quad \frac{7}{9}.$$

These are sometimes termed "vulgar" fractions, and cannot be the type spoken of. Lagrange's reference to "fractions in the system," or "systematic fractions" can only mean those commonly (and quite profanely) called "decimals." Regardless of the number base used, they are of the form:

$$.6825, \quad \text{or} \quad .125, \quad \text{or} \quad .142857\dot{1}4285\dot{7}$$

The last figure above represents the form assumed by the vulgar fraction one-seventh, when expressed as a "decimal" of the Base Ten. The dots above the figures 1 and 7 indicate that this series of figures is a revolving (or circulating,) "decimal," and must be carried out to the required degree of accuracy; but it also indicates that it is inexact because there is always a remainder, and that exactitude can only be secured by reverting to the vulgar type of fraction.

In representing fractions in this way, a revolving "decimal" always results when the number of parts taken is a number that is prime to the base. In the case of Base Eleven, the only fractions that would *not* be revolving "decimals" would be the elevenths, (as: one-eleventh, or five-elevenths,) and multiples thereof.

In order to afford a better concept of the relative convenience of the two bases, we show below a table giving the reciprocals of the numbers two to twelve on Bases Eleven and Twelve:

n	Reciprocals Base Eleven	Base Twelve
2	.555555	.6
3	.373737	.4
4	.282828	.3
5	.222222	.249724
6	.191919	.2
7	.163163	.186X35
8	.141414	.16
9	.124986	.14
X	.111111	.124972
Ɛ	.1	.111111
Twelve	.0X0X0X	.1

A review of the table would lead one to agree with Lagrange's statement that a systematic fraction of the prime base is represented in a unique way, but that this is an advantage is open to some question. The relative convenience of the Twelve Base is quite apparent.

Tobias Dantzig, in his book, "Number, the Language of Science," sets forth these relative views very clearly. With the consent of the publishers, (The Macmillan Co., New York,) we quote from his excellent work.

"It is interesting to speculate what turn the history of culture would have taken if instead of flexible fingers man had had just two 'inarticulate' stumps. If any system of numeration could at all have developed under such circumstances, it would have probably been of the binary type.

"That mankind adopted the decimal system is a *physiological accident.* Those who see the hand of Providence in everything will have to admit that Providence is a poor mathematician. For outside its physiological merit the decimal base has little to commend itself. Almost any other base, with the possible exception of *nine*, would have done as well and probably better.

"Indeed, if the choice of a base were left to a group of experts, we should probably witness a conflict between the practical man, who would insist on a base with the greatest number of divisors, such as *twelve*, and the mathematician, who would want a

prime number, such as *seven* or *eleven*, for a base. As a matter of fact, late in the eighteenth century the great naturalist Buffon proposed that the duodecimal system (base 12) be universally adopted. He pointed to the fact, that 12 has 4 divisors, while 10 has only two, and maintained that throughout the ages this inadequacy of our decimal system had been so keenly felt that, in spite of ten being the universal base, most measures had 12 secondary units.

"On the other hand the great mathematician Lagrange claimed that a prime base is far more advantageous. He pointed to the fact that with a prime base every systematic fraction would be irreducible and would therefore represent the number in a unique way. In our present numeration, for instance, the decimal fraction .36 stands really for many fractions:

$$\frac{36}{100}, \qquad \frac{18}{50}, \qquad \text{and} \qquad \frac{9}{25},$$

Such an ambiguity would be considerably lessened if a prime base, such as eleven, were adopted.

"But whether the enlightened group to whom we would entrust the selection of the base decided on a prime or a composite base, we may rest assured that the number *ten* would not even be considered, for it is neither prime nor has it a sufficient number of divisors."

As a closing comment on the Base Eleven, consider the following numbers, expressed in the notation of that base:

12 21 27 34 38 49

These are all prime numbers, yet we are unable to tell, without actually testing them, not only whether or not they are prime, but, surprisingly, whether or not they are odd or even. Exactly the same condition would apply to any prime base, and to the odd numbers, when used as bases.

Dozenal Quirk: Using duodecimals instead of decimals, it would be possible to save exactly *3X* digits in numbering the pages of *XX* egro *(90%)* of the books published each year.

This is true for all books having from *ƐƐ* pages up to, and including *6Ɛ3* pages. From *6Ɛ4* pages up, an additional digit is saved for each added page until *ƐƐƐ* pages are reached. After this the saving becomes constant at *546* up to *5953* pages.

Harry C. Robert, Jr.

TO CALCULATE THE $\text{LOG}_{10}\ 2$

by Harry C. Robert, Jr.

In these modern times, nearly every one has excellent mathematical reference tables at his elbow. Many of us, including some capable mathematicians, know little or nothing of some of the simple "lost arts" of arithmetic. For instance, how many today realize the neatness of the old simple method of direct computation of logarithms? Yet this requires only an understanding of the elementary laws of exponents and arithmetic. Using the dozenal base, let us calculate $\log_{10}\ 2$:

Let $\log\ 2 = L_0$

Then $(10)^{L_0} = 2$

Obviously $0 < L_0 < 1$,

or $L_0 = 0 + \frac{1}{L_1}$

and $(10)^{0+\frac{1}{L_1}} = 2$

$\div\ (10)^0=1,\quad (10)^{\frac{1}{L_1}} = 2$

and raise to L_1 power, $\quad 10 = (2)^{L_1}$.

Now $3 < L_1 < 4$,

and $L_1 = 3 + \frac{1}{L_2}$

then, $10 = (2)^{3+\frac{1}{L_2}}$

$\div\ (2)^3,\quad \frac{10}{8} = \frac{3}{2} = (2)^{\frac{1}{L_2}}$

and raise to L_2 power, $\quad \left(\frac{3}{2}\right)^{L_2} = 2$.

Now $1 < L_2 < 2$

and $L = 1 + \frac{1}{L_3}$

and $\left(\frac{3}{2}\right)^{1+\frac{1}{L_3}} = 2$

$\div\ \left(\frac{3}{2}\right)^1,\quad \left(\frac{3}{2}\right)^{\frac{1}{L_3}} = \frac{4}{3}$

and raise to L_3 power, $\quad \frac{3}{2} = \left(\frac{4}{3}\right)^{L_3}$

Now $1 < L_3 < 2$

and $L_3 = 1 + \frac{1}{L_4}$

and $\frac{3}{2} = \left(\frac{4}{3}\right)^{1+\frac{1}{L_4}}$

$\div\ \left(\frac{4}{3}\right)^1,\quad \frac{9}{8} = \left(\frac{4}{3}\right)^{\frac{1}{L_4}}$

and raise to L_4 power, $\quad \left(\frac{9}{8}\right)^{L_4} = \frac{4}{3}$.

Now $2 < L_4 < 3$

and $L_4 = 2 + \frac{1}{L_5}$

and $\left(\frac{9}{8}\right)^{2+\frac{1}{L_5}} = \frac{4}{3}$

$\div\ \left(\frac{9}{8}\right)^2,\quad \left(\frac{9}{8}\right)^{\frac{1}{L_5}} = \frac{194}{183}$

and raise to L_5 power, $\quad \frac{9}{8} = \left(\frac{194}{183}\right)^{L_5}$.

Now, obviously, $2 < L < 3$

And the process may be repeated as often as desired to secure any necessary degree of accuracy. However, if we stop here, we may approximate L_5 as follows:

$$L_5 = \frac{\frac{9}{8} - 1}{\frac{194}{183} - 1}\text{, approximately,}$$

$$= 2.4\text{, approximately,}$$

and substituting in the several equations for L_0, L_1, L_2, L_3, L_4, etc., we obtain:

$$L_0 = 0 + \frac{1}{L_1} = 3 + \frac{1}{L_2} = 1 + \frac{1}{L_3} = 1 + \frac{1}{L_4} = 2 + \frac{1}{L_5}$$

$$\text{or,}\quad L_0 = 0 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2.4}}}}}$$

or, approximately, $L_0 = \frac{11.8}{41.0} = 0.341\ Ɛ61$. Rounding this off to four places: $L_0 = \text{Log}_{10}\ 2 = 0.342\ 0$. One more step, L_6, would give us five correct places; etc. The log is shown in six-place tables as $0.342\ 01Ɛ$.

I do not find this method given in any work more recent than Davies, "Cyclopedia of Mathematics," 1856. The approximation of the last term is my idea, although the idea is not at all unusual. Davies does not use it in this case.

FINDING SQUARE ROOTS

by George S. Terry

Probably, not many people enjoy going through the laborious process of extracting the square roots of larger numbers. They will be delighted to learn of an easy method that makes this work unnecessary. They can read the result from a simple little table.

n	$\sqrt{n}$	
3	189	600
4	200	6Ɛ2
5	22Χ	78Ɛ
6	255	85Χ
7	279	920
8	29Ɛ	997
9	300	Χ48
Χ	31Ɛ	ΧƐ5
Ɛ	33Χ	1000

Suppose we need to find $\sqrt{\pi}$. $\pi = 3.184\ 809$. Our number is between 3 and 4. Two values are given for $\sqrt{3}$. We must choose between them, by looking at our number. Its root evidently starts with 1 rather than with 6, so we use the first of the root columns.

$\sqrt{3} = 1.89$ and the difference between this and $\sqrt{4}$ is $2.00 - 1.89 = 0.33$. Multiply this by the fraction in π, namely by .185, and add to the value of $\sqrt{3}$.

We have $\sqrt{\pi} = 1.89 + .185(.33) = 1.93$

Divide π by this approximate root, and we get 1.935 73 . Average this with 1.93 and we get a close value 1.932 97 .

Put more figures on π = 3.184 809 493 Ɛ92 and proceed, if you wish a closer value. Divide this more accurate π by our more approximate root, and we get 1.932 975 819 561 . Average this with 1.932 97, and we get $\sqrt{\pi}$ = 1.932 972 Χ0Χ 890, which is as good a value as you probably need. But you may still continue, if you put more figures on π and proceed as before.

If your n begins with 1 or 2, find $\sqrt{4n}$ and divide the result by 2.

TABLES OF THE ENDINGS OF SQUARES

Decimal and Duodecimal

Decimal Squares

N	0	1	2	3	4	5	6	7	8	9
0	00	01	04	09	16	25	36	49	64	81
1	00	21	44	69	96	25	56	89	24	61
2	00	41	84	29	76	25	76	29	84	41
3	00	61	24	89	56	25	96	69	44	21
4	00	81	64	49	36	25	16	09	04	01
5	00	01	04	09	16	25	36	49	64	81
6	00	21	44	69	96	25	56	89	24	61
7	00	41	84	29	76	25	76	29	84	41
8	00	61	24	89	56	25	96	69	44	21
9	00	81	64	49	36	25	16	09	04	01

N^2 Ends	Last Pair	N
0	00	0
1	01 21 41 61 81	1 9
4	04 24 44 64 84	2 8
5	25	5
6	16 36 56 76 96	4 6
9	09 29 49 69 89	3 7

Duodecimal Squares

N	0	1	2	3	4	5	6	7	8	9	Χ	Ɛ
0	00	01	04	09	14	21	30	41	54	69	84	Χ1
1	00	21	44	69	94	01	30	61	94	09	44	81
2	00	41	84	09	54	Χ1	30	81	14	69	04	61
3	00	61	04	69	14	81	30	Χ1	54	09	84	41
4	00	81	44	09	94	61	30	01	94	69	44	21
5	00	Χ1	84	69	54	41	30	21	14	09	04	01
6	00	01	04	09	14	21	30	41	54	69	84	Χ1
7	00	21	44	69	94	01	30	61	94	09	44	81
8	00	41	84	09	54	Χ1	30	81	14	69	04	61
9	00	61	04	69	14	81	30	Χ1	54	09	84	41
Χ	00	81	44	09	94	61	30	01	94	69	44	21
Ɛ	00	Χ1	84	69	54	41	30	21	14	09	04	01

N^2 Ends	Last Pair	N
0	00	0
	30	6
1	01 21 41 61 81 Χ1	1 5 7 Ɛ
4	04 44 84	2 Χ
	14 54 94	4 8
9	09 69	3 9

Comparison

100 duodecimal squares end in one of 4 figures, or in one of 14 pairs, relatively 4 and 14 egro.

84 decimal squares end in one of 6 figures, or in one of 1Χ pairs, relatively 8.8 and 27.8 egro.

SQUARE SUMS OF CONSECUTIVE SQUARES

by George S. Terry and Harry C. Robert, Jr.

High on the priority list of the Mathematical Research Committee is an item of original research involving the study of those sums of consecutive squares which are themselves square. The brief preliminary paper which follows will serve to introduce this absorbing problem.

An interesting and sometimes useful feature of dozenal numbers is the simplicity of the end figures of their squares. The last figure of a square number has the choice of only four integers dozenally as compared with six in the denary scale; the last pair has the choice of only a dozen and four pairs out of a total of a gross of pairs, as compared with a dozen and ten pairs decimally out of a total of approximately two-thirds of a gross.

This feature has been useful in attempting to answer the question:- What N consecutive squares sum to a square? For example:-

$3^2 + 4^2 = 5^2$ where $N = 2$; Mid-term $= 3\text{-}1/2$

$1^2 + 2^2 + \ldots + 1Ɛ^2 + 20^2 = 6X^2$; $N = 20$; Midtm $= 10\text{-}1/2$

$16^2 + 17^2 + \ldots + 23^2 + 24^2 = 65^2$; $N = Ɛ$; Mid-term $= 1Ɛ$

Not much is known about what number of squares in the run can produce a square sum.

The Sum of such a series is $N\left[a^2 + \frac{N^2 - 1}{10}\right]$ where a^2 is the mid-square. A study of two-figure square endings and of divisibility indicates that no N which ends 3, 5, 6, 7, 8 or X can produce a square sum. This eliminates half the N's we need consider. Of the sums of those remaining, only about one-ninth give possible square endings.

Following are least results for all possible N less than a gross.

N	a	N	a	N	a
41	41	29	1Ɛ	Ɛ	1Ɛ
61	33X			1Ɛ	16
81	53			3Ɛ	3XX
X1	214			4Ɛ	43
				8Ɛ	13733

N	a	N	a	N	a
20	10-1/2	2	3-1/2	74	177-1/2
80	10-1/2	22	31-1/2		
		42	27-1/2		
		62	199-1/2		
		X2	92-1/2		

When N itself is a square as is the case with $N = 41$ and $X1$ in the above table, the number of possible values of "a" is strictly limited, and for those two cases the values given are the only ones. For every case of N not square there are an infinite number of values of "a" for which the sum of N squares is a square. $N = 2$ has one series of solutions: $N = 80$ has eight. Each series has a simple recurrence formula whereby additional values may be obtained from preceding ones. For example:

for $N = 2$; $a_1 = 3\text{-}1/2$; $a_2 = 18\text{-}1/2$; $a_3 = 9Ɛ\text{-}1/3$

or in general, $a_n = 6a_{n-1} - a_{n-2}$

This same formula can be used for all cases where $N = 2k^2$. For other values of N the recurrence formula is similar in form but different coefficients are necessary.

BASIMALS OR RADICIMALS

We are constantly confronted by the need for a proper term to apply to fractions of the systemic type. The public is accustomed to the use of the word "decimals." In connection with the use of the duodecimal base, we dodekaphiles habitually use the term "duodecimals." Neither of these usages is semantically proper. Nor is the use of the expression "decimal-form fractions" more correct in the least degree.

What is desired is an expression for those quantities, less than unity and greater than zero, which are stated in the form of fractions of the base, with the denominator understood. The proper term should be applicable regardless of the scale of notation used. Perhaps the obsolete word "parcials" might become acceptable for this use.

We would like to hear your preferences and any suggestions that you may care to offer.

Harry C. Robert, Jr.

PRIME and FACTOR TABLE, NUMBERS 6 n ± 1 BASE XII

Terminations 01 to 2Ɛ

	-01	-05	-07	-0Ɛ	-11	-15	-17	-1Ɛ	-21	-25	-27	-2Ɛ	
0	Unity								5^2			5 7	0
1	5 25			5 27		7 1Ɛ			11^2		5^2 7		1
2	15^2		5 4Ɛ	11 1Ɛ	7 37	5 51					Ɛ 25	15 17	2
3		17 1Ɛ			5 75		Ɛ 35	5 7 11					3
4		7 6Ɛ	Ɛ 45		17 27		5 7 15			5 Ɛ^2		11 3Ɛ	4
5	7 87	5^2 25		15 37		Ɛ 57			5 105	7 8Ɛ		5 107	5
6	5 125	Ɛ 67	11 57	5^3 7					7 χ7	17 3Ɛ	5 12Ɛ	25 27	6
7			5 7 25			5^2 35	11 67			15 51		7 105	7
8		11 75	17 51		5 175	7 11Ɛ		5^2 3Ɛ	Ɛ 8Ɛ		7 11^2		8
9					7 Ɛ 15	11 85	5 18Ɛ			5^2 45		Ɛ^3	9
χ	Ɛ χƐ	5 15^2				27 3Ɛ		7 Ɛ 17	5 205	11 85		5^2 4Ɛ	χ
Ɛ	5 225	7 16Ɛ	31 37	5 Ɛ 25			7 171				5 15 17		Ɛ
10	7 11 17		5 24Ɛ	31 3Ɛ		5 251		15 87		7 18Ɛ		35 37	10
11				7 1χ5	5 11 25		27 51	5 277	7 1χ7		Ɛ 125		11
12		37 3Ɛ	7 15^2			17 8Ɛ	5 Ɛ 31		11 111	5 2χ1	1Ɛ 75	7 205	12
13		5 301	Ɛ 145	11 11Ɛ	35 45	7 21Ɛ		31 4Ɛ	5 17 1Ɛ	Ɛ 147	7 221	5 307	13
14	5 325			5 327	7 237	Ɛ 157	1Ɛ 85	11 12Ɛ	15 Ɛ5		5 32Ɛ		14
15	27 67	Ɛ 167	5 34Ɛ		1Ɛ 8Ɛ	5 15 25		7 255			31 57	11 13Ɛ	15
16		7^2 45	1Ɛ 85	17 Ɛ5	5 375		7 271	5 377			37 51	31 5Ɛ	16
17	7 15 1Ɛ		11 157	35 57			5 17 25	27 75	Ɛ 18Ɛ	5 7 67		15 117	17
18	37 57	5 401		7^2 4Ɛ	Ɛ 19Ɛ		11 167		5 7 6Ɛ		35 5Ɛ	5 Ɛ 45	18
19	5^2 Ɛ^2	11 175	7 301	5 427			15 12Ɛ	Ɛ 1Ɛ1		37 5Ɛ	5 11 3Ɛ	7 17 1Ɛ	19
1χ		17 11Ɛ	5^2 χ7	Ɛ 15^2		5 7^2 11			27 87	1Ɛ Ɛ7	7 321		1χ
1Ɛ		27 8Ɛ			5^2 7 17			5 1Ɛ 25	3Ɛ 5Ɛ	11 195			1Ɛ
20						1Ɛ 107	5^2 Ɛ7	7^2 5Ɛ	4Ɛ^2	5 15 35	Ɛ 225		20
21	11 1Ɛ1	5 7 87		1Ɛ 111			7 Ɛ 3Ɛ		5^3 25	17 13Ɛ		5 507	21
22	5 7 8Ɛ	1Ɛ 117	Ɛ^2 27	5 527	11 15^2		45 5Ɛ			7^3 Ɛ	5^2 107		22
23		15 171	5 17 35	7 3χ5	3Ɛ 6Ɛ	5 Ɛ 5Ɛ			7 11 37				23
24	31 81	Ɛ 267	7 401	11 21Ɛ	5 575			5 577		27 χƐ	15 17Ɛ	7^2 6Ɛ	24
25		31 95	3Ɛ 75	45 67	4Ɛ 5Ɛ	7 41Ɛ	5 59Ɛ	11 15 17		5 25^2	7 421		25
26	25 105	5^4 125		51 5Ɛ	7 437			37 85	5 Ɛ 67		17 171	5 11 57	26
27	5 17 3Ɛ	35 91	15 19Ɛ	5^2 12Ɛ	Ɛ^2 31			7 455	57^2		5 25 27	Ɛ 2χ1	27
28	Ɛ 2χƐ	7 46Ɛ	5 11 5Ɛ	27 105		5^3 31	7 471	Ɛ 2Ɛ1	35 95				28
29	7^2 81	57 5Ɛ		Ɛ 301	5 675	17 18Ɛ	11 267	5^2 13Ɛ	15 1Ɛ5	7 48Ɛ			29
2χ	4Ɛ 6Ɛ	11^2 25		7 4χ5		15^3	5 69Ɛ		7 17 37	5^2 145	11 277		2χ
2Ɛ	5Ɛ^2	5 701	7^2 87		27 117	11 285		51 6Ɛ	5 705	31 Ɛ5	Ɛ 325	5^2 7 25	2Ɛ

Harry C. Robert, Jr.

PRIME and FACTOR TABLE, NUMBERS 6 n ± 1 BASE XII

Terminations 31 to 5Ɛ

	-31	-35	-37	-3Ɛ	-41	-45	-47	-4Ɛ	-51	-55	-57	-5Ɛ	
0					7^2		5 Ɛ			5 11			0
1		5 31	Ɛ 15					7 25	5 35	Ɛ 17		5 37	1
2	5^2 11	7 3Ɛ		5 57		Ɛ 27	7^3				5 5Ɛ		2
3	7 57	Ɛ 37	5^2 17		11 31	5 81			15 25	7 5Ɛ			3
4				7 75	5^4	15 31		5 χ7	7^2 11				4
5			7 91	11 4Ɛ			5^2 27	17 35	Ɛ 5Ɛ	5 111		7 85	5
6	15 45	5 131			Ɛ 6Ɛ	7 χƐ		11 5Ɛ	5^2 31		7^2 17	5 Ɛ 15	6
7	5 Ɛ 17			5 157	7 107			Ɛ 81		25 31	5^2 37	11 6Ɛ	7
8	25 35		5 17Ɛ	Ɛ 91		5 181	15 5Ɛ	7 125			1Ɛ 45		8
9	27 37	7 13Ɛ	11 87	15 67	5 1χ5	17 5Ɛ	7 141	5 1χ7	1Ɛ 4Ɛ		25 3Ɛ		9
χ	7 157						5 11 1Ɛ		17 67	5 7 37	Ɛ Ɛ5		χ
Ɛ		5^3 11		7 175	1Ɛ 5Ɛ		Ɛ 105	27 45	5 7 3Ɛ	15 81	11 χ7	5 237	Ɛ
10	5 255	25 51	7 Ɛ 1Ɛ	5^2 5Ɛ		11 Ɛ5				Ɛ 117	5 25Ɛ	7 195	10
11	1Ɛ 6Ɛ		5 27Ɛ	17 85	15 95	5^2 7 Ɛ	35 3Ɛ			11 105	7 1Ɛ1	25 57	11
12		Ɛ^2 15	25 5Ɛ		5 7 4Ɛ		17 91	5^2 6Ɛ	27 57				12
13	11^3	27 5Ɛ			3Ɛ^2		5 30Ɛ	7 225		5^2 75	15 χƐ	1Ɛ 81	13
14		5 7 57			11 131		7 241	15 Ɛ7	5 Ɛ 37	1Ɛ 87		5^3 17	14
15	5 7 5Ɛ	17 χƐ	3Ɛ 45	5 357	Ɛ 16Ɛ	35 51		1Ɛ 91	11 141	7 25Ɛ	5 35Ɛ	Ɛ 171	15
16	Ɛ 17Ɛ		5 15 27	7 11 25	17 Ɛ7	5 1Ɛ^2		Ɛ 181	7 277				16
17	3Ɛ 4Ɛ		7 291	Ɛ^2 1Ɛ	5 3χ5			5 11 37				7 295	17
18		1Ɛ χ7	31 67		25 85	7 2χƐ	5 40Ɛ		15 125	5 17 27	7 2Ɛ1	11 16Ɛ	18
19		5 431		31 6Ɛ	7 307	15 131			5 435		Ɛ 1Ɛ5	5 437	19
1χ	5 455		11^2 17	5 457			Ɛ 205	7 325		45 51	5 45Ɛ	35 67	1χ
1Ɛ	15 145	7 33Ɛ	5 Ɛ 51			5 481	7 11 31			Ɛ 217	27 91	15 147	1Ɛ
20	7 357	11 1χ5		27 95	5 4χ5	Ɛ^2 25		5 17 31		7 35Ɛ	11 1χ7		20
21		Ɛ 237		7 375	35 75	11 1Ɛ5	5 15 37		7 377	5 511	17 141		21
22	17 147	5 531	7 391	15 167			25 χƐ		5 535	11 205	31 87	5 7 91	22
23	5^2 111			5 557	27 χ7	7 3χƐ			Ɛ 25Ɛ	4Ɛ 57	5 7 95	31 8Ɛ	23
24	11 221		5^2 117		7 Ɛ 45	5 17 37	51 57			15 181		Ɛ 271	24
25	Ɛ 27Ɛ			35 87	5^2 11^2			5 7 Ɛ^2	17 167			27 Ɛ5	25
26		7^2 75		Ɛ 291	15 195		5^4 7	25 107	11 241	5 611	35 8Ɛ		26
27	7 457	5 15 45		11 24Ɛ					5^2 131	7 45Ɛ	1Ɛ 145	5 637	27
28	5 655			5 7^2 17		4Ɛ 67		11 25Ɛ	7 1Ɛ 25		5^2 Ɛ 15		28
29			5 7 Ɛ5			5 27^2	Ɛ 17 1Ɛ	15 1Ɛ7			51 67	7 11 45	29
2χ			Ɛ 315		5 1Ɛ 37	7^2 85		5 6χ7		Ɛ^2 35	7 4Ɛ1		2χ
2Ɛ			11 15 1Ɛ		7 507	Ɛ 327	5 70Ɛ			5 711		17 1χ5	2Ɛ

Harry C. Robert, Jr.	PRIME and FACTOR TABLE, NUMBERS 6 n ± 1 BASE XII											Terminations 61 to 8Ɛ	
	-61	-65	-67	-6Ɛ	-71	-75	-77	-7Ɛ	-81	-85	-87	-8Ɛ	
0		7 Ɛ			5 15		7 11	5 17					0
1	7 27	11 15					5 3Ɛ			5 7^2	11 17		1
2	17^2	5 61		7 45		11 25			5 7 Ɛ		15 1Ɛ	5 67	2
3	5 85		7 61	5 87	Ɛ 3Ɛ			15 27	$1Ɛ^2$	11 35	5 8Ɛ	7^2 Ɛ	3
4	Ɛ 4Ɛ		5 XƐ			5 7 17	1Ɛ 25	Ɛ 51			7 81		4
5	11 51		15 3Ɛ	Ɛ 61	5 7 1Ɛ			5 117	17 37				5
6			1Ɛ 35		11 61		5 13Ɛ	7 Ɛ5	27^2	5 141			6
7	1Ɛ 3Ɛ	5 7 27					7 111		5 11 15		Ɛ 85	5 167	7
8	5^2 7^2			5 11 17		15 61	Ɛ 95	25 37		7 12Ɛ	5 18Ɛ		8
9	31^2		5^3 Ɛ	7 145		5 1Ɛ1	17 61	11 8Ɛ	7 147	Ɛ X7		1Ɛ 51	9
X	15 75	31 35	7^2 27		5^2 51	Ɛ Ɛ7		5 217	25 45	1Ɛ 57		7 11 15	X
Ɛ		Ɛ 107				7 17Ɛ	5^2 57	1Ɛ 61	35^2	5 241	7 181	17 75	Ɛ
10		5 17^2	11 Ɛ7		7^2 31	1Ɛ 67	15 8Ɛ		5^2 61	27 4Ɛ		5 267	10
11	5 285			5 15 1Ɛ	17 87	31 45	11 107	7 1Ɛ5	Ɛ 12Ɛ		5^2 67		11
12		7 11 1Ɛ	5 2XƐ		Ɛ 13Ɛ	5 2Ɛ1	7^2 37			25 61	11 117	Ɛ 141	12
13	7 Ɛ 25				5 315	11 125		5 Ɛ 35	31 51	7 15 17	27 61		13
14				7 Ɛ 27			5 33Ɛ		7^4	5 11 31	25 6Ɛ		14
15		5^2 85	7 17^2		15 105	37 4Ɛ			5 365			5 7 61	15
16	5 11 35	15 111		5^2 8Ɛ		7 27Ɛ					5 7^2 Ɛ		16
17	45^2	25 81	5 3XƐ		7 11 27	5^2 95	Ɛ 195	17 105			15 11Ɛ		17
18			Ɛ 1X5		5 415			5^2 7 15	11 171	Ɛ 1X7	17 111	25 87	18
19	17 117	7 30Ɛ	25 8Ɛ	11 17Ɛ		Ɛ 1Ɛ7	5 7 75			5^5	45 4Ɛ	27 85	19
1X	7 327	5 Ɛ 4Ɛ	15 13Ɛ					11 18Ɛ	5 465	7 32Ɛ		5^2 XƐ	1X
1Ɛ	5 485			5 7 81	37 67	17 12Ɛ	35 6Ɛ		7 347		5 48Ɛ	11 19Ɛ	1Ɛ
20			5 7 85			5 4Ɛ1		45 57	Ɛ 15 17			7 365	20
21			11 1Ɛ7	25 X7	5 Ɛ 57	7 15 27		5 517			7 $1Ɛ^2$	Ɛ 241	21
22	Ɛ 24Ɛ			37 75	7 397		5 11 4Ɛ	Ɛ 251	1Ɛ 11Ɛ	5 541			22
23	15 175	5 11 51		Ɛ 17^2	25 Ɛ5	35 81	1Ɛ 125	7 3Ɛ5	5 565		11 217	5 15 3Ɛ	23
24	5 585	7 40Ɛ		5 587	1Ɛ 12Ɛ	11 225	7 17 27				5 58Ɛ		24
25	7 427		5 1Ɛ 31			5 5Ɛ1	15 18Ɛ			7 11 3Ɛ	Ɛ 285		25
26	1Ɛ 13Ɛ		45 6Ɛ	7 15 31	5 615		Ɛ 295	5 617	7 447			17 175	26
27	11 251	17 17Ɛ	7 Ɛ 4Ɛ			25 111	5 63Ɛ	3Ɛ 81		5 Ɛ 6Ɛ		7 465	27
28	27 107	5 661	37 91		11 17^2	7 Ɛ 51	31 X7		5 665	15 1Ɛ1	7 481	5 1Ɛ 35	28
29	5^2 141	Ɛ 307		5 687	7 497	3Ɛ 87	25 11Ɛ	31 XƐ	11 271	1Ɛ 157	5 68Ɛ	37 95	29
2X			5^2 147	11 27Ɛ	15 205	5 6Ɛ1		7 1Ɛ 27		17 19Ɛ			2X
2Ɛ		7 15 37		3Ɛ 91	5^3 35	1Ɛ 167	7 511	5 11 67	Ɛ 32Ɛ	45 81	31 Ɛ7		2Ɛ

Harry C. Robert, Jr.	PRIME and FACTOR TABLE, NUMBERS 6 n ± 1 BASE XII											Terminations 91 to ƐƐ	
	-91	-95	-97	-9Ɛ	-X1	-X5	-X7	-XƐ	-Ɛ1	-Ɛ5	-Ɛ7	-ƐƐ	
0			5 1Ɛ	7 15	$Ɛ^2$	5^3			7 17			Ɛ 11	0
1	Ɛ 1Ɛ		7 31		5 45			5^2 Ɛ				7 35	1
2			11 27	Ɛ 31		7 4Ɛ	5 6Ɛ			5^2 15	7 51		2
3		5 91		17 25	7 67		11 37		5 95			5^2 1Ɛ	3
4	5 Ɛ5	11 45		5 Ɛ7	15 35		17 31	7 85		1Ɛ 27	5 Ɛ 11		4
5		7^2 15	5 11Ɛ		25^2	5 11^2	7 $Ɛ^2$	1Ɛ 31					5
6	7 Ɛ7		Ɛ 75		5 145	1Ɛ 37		5 147		7 Ɛ 11	15 4Ɛ	17 45	6
7		17 4Ɛ		7^2 1Ɛ		Ɛ 87	5 16Ɛ	15 57	7 117	5 171	27 31		7
8	11 81	5 Ɛ 1Ɛ	7 131	27 35	17 57				5 195			5 7 31	8
9	5 1Ɛ5		15 6Ɛ	5 1Ɛ7	11 91	7^2 25					5 7 35		9
X			5 21Ɛ		7 167	5 221			$Ɛ^2$ 11	17 6Ɛ			X
Ɛ				11 XƐ	5 Ɛ 27		25 4Ɛ	5 7^3	15 85			Ɛ 111	Ɛ
10	Ɛ 11Ɛ	7 19Ɛ	17 81		37^2	15 91	5 7 45	Ɛ 11^2		5 271			10
11	7 1Ɛ7	5 291		Ɛ 131					5 295	7^2 35		5 11 27	11
12	5^3 15			5 7 51				17 95	7 217		5 2ƐƐ	15 X7	12
13			5^2 7 11	37 45		5 321		25 67			$Ɛ^2$ 17	7^2 3Ɛ	13
14	17 X7		35 4Ɛ		5^2 81	7 24Ɛ	Ɛ 11 15	5 347			7 251		14
15		11 145	Ɛ 175	15 107	7 267	27 6Ɛ	5^2 87		25 75	5 Ɛ 3Ɛ	11 147		15
16	31 61	5 391				Ɛ 11 17		7 285	5^2 91			5 397	16
17	5 3Ɛ5	7 Ɛ 31		5 3Ɛ7			7 2X1	3Ɛ 51	17 107	11^2 15	5^3 1Ɛ		17
18	7^2 51	35 61	5 41Ɛ			5 421	27 81		1Ɛ XƐ	7 2ƐƐ			18
19	11 181		37 61	7 315	5 15 31	3Ɛ 57	1Ɛ Ɛ5	5 447	7 Ɛ 35	25 91			19
1X	25 95	15 141	7^2 57	17 125	Ɛ 11 1Ɛ	31 75	5 46Ɛ			5 471		7 Ɛ 37	1X
1Ɛ	Ɛ 21Ɛ	5^2 Ɛ5	1Ɛ 105	3Ɛ 61		7 34Ɛ	17 131	Ɛ 221	5 11 45		7 15 25	5 497	1Ɛ
20	5 1Ɛ 27	37 6Ɛ		5^2 Ɛ 11	7^2 61			15 157	31 81		5 4ƐƐ	4Ɛ 51	20
21		3Ɛ 67	5 51Ɛ		51^2	5^2 105		7 11 35		31 85		17 145	21
22		7 17 25	15 16Ɛ		5 545	45 61	7^2 67	5^3 27			Ɛ 255	11^2 1Ɛ	22
23	7 3Ɛ7				17 157		5 Ɛ 61			5^2 7 1Ɛ		25 Ɛ7	23
24	35 85	5 591	Ɛ 11 25	7 415				1Ɛ 131	5 7^2 15	Ɛ 277	37 81	5^2 11Ɛ	24
25	5 5Ɛ5		7 431	5 5Ɛ7		Ɛ 15 1Ɛ	11 237	4Ɛ 61	27 Ɛ7	17 16Ɛ	5 5ƐƐ	7 435	25
26	37 87	Ɛ 11 27	5 61Ɛ	1Ɛ 141		5 7 X7			51 61		7^3 11		26
27	15 1X5	1Ɛ 247	17 181		5 7 XƐ	11 255		5 647		37 8Ɛ		15 1X7	27
28	45 75			25 117			5 66Ɛ	7 485	Ɛ 2ƐƐ	5 11 61	3Ɛ 85		28
29		5 7 Ɛ7	27 111		Ɛ 30Ɛ		7 15 35	17 195	5 695		57 61	5 Ɛ 75	29
2X	57 Ɛ 11			5 15 4Ɛ	25 125			Ɛ 321	3Ɛ 8Ɛ	7 4ƐƐ	5 17 45		2X
2Ɛ	17 1X7		5 71Ɛ	7 Ɛ 57	11 291	5 721			7 517	27 11Ɛ		5Ɛ 61	2Ɛ

NOTES ON THE ARABIC NUMERALS

We have extracted the following notes on the introduction of the Arabic numerals into Europe from The Cyclopedia of Education, published by Wm. Swan Sonnenschein in London, 1889.

"The Arabic numerals, which have had so much to do with the progress of arithmetic in the western world, appear to have been known to the Hindoos as early as the fifth century. They were certainly introduced by the Arabians into Spain, though the precise date is not known. Gerbert, afterwards Pope Sylvester II, who died in 1003, is said to have carried these numerals from the Moors of Spain into France in 960, though this is doubted by many.

"The more probable account is that Leonard of Pisa introduced them in 1202 in his Liber Abbaci. Others have supposed that the Alonsine tables (first constructed by the Moors at the court of Alonso,) first contained this system. Certain it is that this system was in the hands of the Persians and the Arabs before the twelfth century, and that they ascribed it to the Hindoos.

"There seems to have been no general use of Arabic numerals in Europe before the invention of printing, and the works of Caxton do not contain them except as woodcuts. Merchants continued to keep their accounts in Roman numerals until the sixteenth century.

"After the discovery of the cipher, there were several courses open, and we might have had a binary, quinary or duodenary scale instead of the decimal notation."

QUERY

What numbers are equal to the sum of powers of their integers?

In squares $25 = 2^2 + 5^2$ $\quad X5 = X^2 + 5^2$

In cubes $668 = 6^3 + 6^3 + 8^3$ $\quad 577 = 5^3 + 7^3 + 7^3$

$X83 = X^3 + 8^3 + 3^3$ $\quad 11XX = 1^3 + 1^3 + X^3 + X^3$

Will someone tell me if this list is complete?

George S. Terry

PERIODS OF PRIMES

by Ralph H. Beard

In a recent letter, Charles Q. De France remarked that: "Many of the smaller primes appear in pairs, as (decimally,) 41 and 271 with 5-place reciprocals, 7 and 13 with 6 places, 329 and 4649 with 7, 73 and 137 with 8, and so on." He tried the result of writing six 7's in a row, and found the following factors:

7)	777 777
7)	111 111
3)	15 873
11)	5 291
13)	481
37)	37

This led him into further checking, and he found that: "Repetition of any member of a pair, or group, of primes as many times as there are places in their reciprocals, will form a composite number divisible twice by that prime and by the companion."

Analysis of this situation in regarding Base Twelve finds the same performance. But Mr. De France's statement should be that:

> Any prime, that is prime to the number-base used, is a factor of the composite number formed of a series of as many 1's as there are places in its revolving reciprocal.

Where primes are factors of the number base, this will not be valid. This can be readily checked by noting the absence of 5 in Mr. De France's figures, and, for the Twelve-Base, that *3* is not a factor of any number ending in *1*.

Where the periods of reciprocals of primes are factors of the periods of the reciprocals of other primes, then they are also factors of the other's larger series of *1*'s. For instance, *11* has a *2*-place reciprocal, *111* has a *3*, and *7* and *17* have *6*'s. Therefore, *7*, *11*, *17*, and *111* are all factors of *111 111*.

7)	*111 111*
11)	*1X 537*
17)	*1 887*
111)	*111*

This particular congruence directs our scrutiny to other congruences existing between primes, their reciprocals, and the periods and submultiples of the reciprocals. Congruences are, of course, invalid on different number bases. But the analysis of numbers for primes is mainly a matter of congruences and residues.

The binary base separates the multiples of 2 from the odd numbers by congruences, and affords the classification of primes into the form of $4n \pm 1$. Base 6 eliminates the odd multiples of 3 and permits the refined classification of the form of $6n \pm 1$. The 10-Base serves to facilitate the elimination of the multiples of 5.

With the application of the Base Twelve, something new has been added. On this base, all primes end *1*, *5*, *7*, or *Ɛ* . And these terminations each identify classes of primes with different characteristics.

Since George Terry's Table of the Periods and Submultiples of the Reciprocals of Primes is of immense value in these considerations, it is again reprinted here, with his consent and the consent of the publishers of his work, "The Dozen System," (Longmans, Green & Co.,) in which it appears.

Mr. Terry has pointed out many of these relationships in that study, and in his "Duodecimal Arithmetic." Among his announcements, are two of outstanding importance. He shows that, on the duodecimal base, the last figure of the prime and the last figure of the reciprocal add up to *10*. For instance, the reciprocal of *7* is *186 X35*. So far as our explorations extend, this is peculiar to the duodecimal base.

Another individuality of the duodecimal base, announced by Mr. Terry, is that: Those primes whose reciprocals extend for the full period, (p - *1*,) end in *5* or *7* only. Consult the table and note the relative frequency of the primes of full period among those ending in *5* and those ending in *7*. Also note that, as stated by F. Emerson Andrews in "New Numbers", the product of multiplying the period by the submultiple equals p - *1*. This product will end in *0*, *4*, *6*, or *X* for primes ending *1*, *5*, *7*, or *Ɛ* respectively. The table brings out these special individualities with beautiful clarity.

Since the primes ending in *5* and *7* are of the form *10*n $\pm$ *5*, (or, if you prefer, *10*n $\pm$ *7*,) the duodecimal base can be credited with another important advantage, adding one more to the long list of ways in which it facilitates numerical operations by fitting into the natural patterns of numbers. It is a refined instrument for expediting research.

PERIODS AND SUBMULTIPLES OF RECIPROCALS OF PRIMES

From The Dozen System by George S. Terry

P.	Per.	S.M.	P.	Per.	S.M.	P.	Per.	S.M.	P.	Per.	S.M.
11	2	6	5	4		7	6		Ɛ	1	X
31	9	4	15	14		17	6	3	1Ɛ	Ɛ	2
51	13	4	25	4	7	27	26		3Ɛ	1Ɛ	2
61	30	2	35	34		37	36		4Ɛ	25	2
81	14	6	45	44		57	56		5Ɛ	2Ɛ	2
91	46	2	75	8	Ɛ	67	22	3	6Ɛ	35	2
			85	84		87	86		8Ɛ	45	2
			95	94		X7	X6		XƐ	55	2
			Ɛ5	Ɛ4		Ɛ7	Ɛ6				
111	3	44	105	104		107	106		11Ɛ	6Ɛ	2
131	76	2	125	124		117	116		12Ɛ	75	2
141	20	8	145	144		147	56	3	13Ɛ	7Ɛ	2
171	96	2	175	8	25	157	12	13	16Ɛ	95	2
181	X0	2	195	194		167	166		17Ɛ	9Ɛ	2
1Ɛ1	Ɛ6	2	1X5	1X4		1X7	46	5	18Ɛ	X5	2
			1Ɛ5	1Ɛ4		1Ɛ7	1Ɛ6		19Ɛ	XƐ	2
221	66	4	205	204		217	86	3	21Ɛ	10Ɛ	2
241	120	2	225	224		237	92	3	24Ɛ	125	2
251	73	4	255	254		267	266		25Ɛ	12Ɛ	2
271	27	10	285	284		277	276		27Ɛ	13Ɛ	2
291	83	4	295	294					2XƐ	155	2
2X1	150	2							2ƐƐ	37	X
2Ɛ1	26	12									
301	90	4	315	314		307	102	3	30Ɛ	165	2
321	170	2	325	324		327	10X	3	32Ɛ	175	2
391	1X6	2	365	364		347	46	9	33Ɛ	17Ɛ	2
			375	34	11	357	11X	3	34Ɛ	2Ɛ	12
			3X5	3X4		377	376		35Ɛ	18Ɛ	2
			3Ɛ5	3Ɛ4		397	396		3XƐ	1Ɛ5	2
						3Ɛ7	3Ɛ6				
401	100	4	415	414		427	426		40Ɛ	205	2
421	63	8	435	434		437	436		41Ɛ	20Ɛ	2
431	109	4	455	454		447	446		45Ɛ	22Ɛ	2
471	13	38	465	464		457	15X	3	46Ɛ	7	7X
481	24	20	485	44	11	497	496		48Ɛ	245	2
4Ɛ1	9X	6	4X5	4X4					4ƐƐ	25Ɛ	2
511	266	2	535	534		507	182	3	51Ɛ	45	12
531	276	2	545	544		517	516		58Ɛ	2X5	2
541	54	10	565	564		527	526		59Ɛ	2XƐ	2
591	3X	16	575	574		557	556		5ƐƐ	2ƐƐ	2
5Ɛ1	159	4	585	584		577	116	5			
			5Ɛ5	5Ɛ4		587	1XX	3			
						5Ɛ7	66	Ɛ			

THE DeFRANCE ALGORITHM FOR FINDING THE LENGTH OF PERIOD OF RECIPROCALS OF PRIMES

By Research Committee

Feeling that some of our members might wish some explanation of the algorithm given in the DeFrance article on page *19* in the Bulletin of February 1947, the Research Committee has prepared the following simple explanation. While the basic theory will be found in most works on number theory under the discussion of congruences, none of these works give such a neat application of the principles as Mr. DeFrance has contrived. No understanding of congruences is required to follow or use this algorithm since it involves only the use of remainders resulting from ordinary division operations.

To find the length of the period of the reciprocal of a prime such as *1X7* we need to know at what place in the division the remainder is *1*. We know this will occur at place *1X6* but it may also occur at some sub-multiple of *1X6*. We also know that since our number ends *7*, the sub-multiple will be odd, the length of period singly even. (See The Dozen System, p. *35*.) $1X6 = 2 \cdot 3 \cdot 5 \cdot 9$; so the length of period may be *1X6*, *76*, *46*, *26*, *16*, *X*, *6* or *2*.

Please, now open your copy of the last Bulletin to page *19*. Since *1,000* divided by *1X7* yields a quotient of *.006* and a remainder of *86*, it is evident that the remainder (R_1) after place *1* is *10* and R_2 after place *2* is *100*. Then by the DeFrance theorem

$$R_3 = \frac{R_1 \cdot R_2}{1X7} \equiv 86 \qquad R_6 = \frac{R_3 \cdot R_3}{1X7} = \frac{(86)^2}{1X7} \equiv 8X$$

$$R_X = \frac{R_1 \cdot R_3 \cdot R_6}{1X7} = \frac{10 \cdot 86 \cdot 8X}{1X7} \equiv 152 \qquad R_{10} = \frac{R_6 \cdot R_6}{1X7} \equiv X5 \qquad R_{16} = \frac{R_{10} \cdot R_6}{1X7} \equiv 182$$

so the length of the period is not *2*, *6*, *X* or *16*.

$$R_{26} = \frac{R_{16} \cdot R_{10}}{1X7} \equiv 121 \qquad R_{46} = \frac{R_{26} \cdot R_{10} \cdot R_{10}}{1X7} = \frac{121(X5)^2}{1X7} \equiv 1$$

and the length of *period is 46 places.*

$$\text{In general} \quad R_{a+b} \equiv \frac{R_a \cdot R_b}{N} \quad \text{and} \quad R_{a+b+c} \equiv \frac{R_a \cdot R_b \cdot R_c}{N}, \ \text{etc.}$$

The foregoing algorithm is entirely general for all primes and with some restrictions can be applied to composite numbers also. It is applicable to any number base and can be used for odd and even lengths of period. For prime numbers with an even length of period the process can be shortened. For example, primes ending *5* or *7*, having an even length of period, this period is in two complementary halves, e.g., *2497* and the remainder at the half-period will be (N-*1*), in the above example *1X6*. So we may test for half-periods *1*, *3*, *5*, *9*, *13*, *23*, *39*, *E3* till we get the remainder *1X6*. $R_1 = 10$; $R_3 = 86$; $R_5 = 46$; $R_9 = 183$; $R_{13} = 11$; and $R_{23} = 1X6$.

In addition to numbers ending *5* or *7*, the foregoing short-cut applies to those primes ending *1* with an even length of period. Note that, for prime numbers having an even length of period, not only are the half-period and full period remainders complementary to N but at each corresponding place in first and second half-

periods the remainders are complementary to N. So when we find two remainders whose sum is N, the difference of their places is the half-period. Thus in the foregoing we find that $R_{26} + R_3 = 121 + 86 = 1X7 = N$, from which we can deduce by $26 - 3 = 23$ that the length of the half-period is *23*.

Only one feature of the more complicated case for composite numbers will be mentioned. If the remainder at the (N-*1*)th place is not *1*, N must be composite.

MATHEMATICAL RECREATIONS

Mary Lloyd, Editor

These mathematical amusements seem to be gaining in popularity and that makes me very happy. Among us dozeners, there are many who are mainly interested in the idea of counting by dozens, and it is well for these people to get out a pencil, once in a while, and give their duodecimal arithmetic a workout. These puzzles serve that purpose well since there is a pleasure in solving any problem.

There is a story about the Fibonacci problem in the last issue. Frederick the Great visited Pisa in 1225, and to test Leonardo's skill, challenged him to find that perfect square which remains a perfect square when increased or decreased by 5. Fibonacci found:

$$3.5^2, \qquad 3.5^2 - 5 = 2.7^2, \qquad \text{and} \qquad 3.5^2 + 5 = 4.1^2$$

Of course, Fibonacci stated his results in vulgar fractions, and in the decimal notation, as we have yet to find that he had any concept of the possibility of any other. But this is just another illustration of the ease with which duodecimals fit the patterns of numbers.

Maurice Kraitchik, in his Mathematical Recreations, gives the general handling of this problem and states that "it was fortunate that Frederick II did not give Fibonacci 1, 2, 3, or 4, instead of 5, for the problem would then be insoluble."

Indirectly, there is a mathematical paradox insinuated here. George Terry says that the only equal regular polygons that can cover a plane are those whose sides number some factor of the dozen; i.e., *3*, *4*, or *6* sides. Also, that when more than one kind of regular polygon is used in these tessellations, only those can be used whose sides number some factor of two dozen; i.e., *3*, *4*, *6*, *8*, or *10* sides.

So here's a problem. Which of the five regular polyhedra can solidly fill space?

The five polyhedra are the tetrahedron, the cube, the octohedron, the dodecahedron, and the icosahedron. It does not require a second thought to select the cube as one of the space-filling forms. But what others? We know that, in normal piling of spheres, for instance, there are twelve surrounding and touching a thirteenth. When these are compressed so as to deform, and completely fill space, what regular solid results?

The puzzles, this time, have some dangerous pitfalls. In the first one, only *X* letters are used and you'll have to watch your step. Just for fun, I picked words in darker hues for these arithmo-crypts. I fudged 'em from the word list of a skeleton puzzle. Don't look now, but the answers will be found on page 24.

Mary.

```
              S J                        Y E D
E M O J ) D T N O J S    M O C H A ) A N A L I D E
          D N T E J                    M O C H A
            O E T A S                  Y C R M E D
            O E L S A                  Y C D O N H
                J O S                      R D H A E

        L A M P
  x       G E S                  B O N E G R I T
      L M Y Y P            +   T O R C C C I L
    L U B G P                E E N I N E E A E
  D O G P
  Y S U M Y P
```

THE MAIL BAG

The first fruits of the creation of the Committee on Mathematical Research appear in this issue, with the publication of the article on the Sums of Consecutive Squares. We wish to express our congratulations.

One of the advantages of the dozen base that is not well known is that it affords close abbreviations of the square roots of numbers with surprising accuracy. The roots of eight out of the first dozen numbers can be closely approximated in two places of duodecimals.

Outstanding among them is the increasingly important $\sqrt{5}$. This root can be stated as *2.2X*, and the closeness of the approximation will be realized by noting that the square of *2.2X* is *5.0004*. Some of the elements of the importance of this figure lie in its place in the equations for the functions of the angles of the pentagon and the dodecahedron, in the summation of the terms of the Fibonacci Series, and in the applications of the Golden Section, or the Golden Mean. The laws of Dynamic Symmetry, promulgated by Jay Hambidge, involve its frequent use.

. . . One of the good friends of The Duodecimal Society is Juan de Dios Tejada of Havana, Cuba, who writes a column called La Marca de Tecnica for the Havana daily Informacion. In his column for 21 March 1947, he devoted the entire column to the Fibonacci Series. He made extensive reference to the material of Harry Robert's article on that subject in the last issue of The Bulletin, with most courteous citation of The Duodecimal Society.

. . . Jorge Carreras, our member who enlisted Mr. Tejada's interest, has now returned to his home in Havana, after a course of studies in American machine techniques under the auspices of the importing house of Feito y Cabezon, of Havana. On his way south to fly to Cuba, he stopped off in New York, and left with us a translation of Mr. Andrews' "An Excursion in Numbers," into Spanish. We hope to have the Spanish edition reprinted for the stimulation of interest in duodecimals among our South American and Central American neighbors.

. . . As a result of the recent publicity on the Society's Annual Award, we were invited to address the Men's Club of the Oakwood Heights Community Church of Staten Island. The group of approximately twenty five responded with gratifying interest. Their requests for copies of The Dozen System, by Mr. Terry, exceeded our available supply, and some had to be mailed later.

. . . The Staten Island Advance carried the announcement of the Award, and a report of the meeting. Notices of the Award were also carried by the N.Y. World Telegram, and The Telephone Review, among a list of others which included the home town newspaper, The York (Pa.) Dispatch.

. . . We have received more of Charles De France's interesting material on "Expansible Integers." Mr. De France has explored this selected field extensively, and has discovered many amazing characteristics of numbers of this type. All of his work has been done on the decimal base, and the labor of developing the corresponding relationships for the duodecimal base, and of testing and checking the resulting figures, requires more time than Ye Ed. has. We would like to hear from a volunteer from among our members who have been interested in Mr. De France's discoveries, for collaboration.

. . . Louis Carl Seelbach is another of our personal stimulators. In his book-worming, he turns up many an interesting line of thought, and gives us the needle. We get absorbed in running this stuff down, and - Blooie! goes the schedule. His latest is the suggestion of a comparative analysis of the possibilities in the game of African Golf, (popularly termed "Craps,") as expressed in the different number bases, two, six, eight, ten, and twelve. They tell us that the Navy can supply considerable material on the subject. Research is always absorbing.

. . . Paul Van Buskirk has been waging doughty war with his keen duodecimal blade among the advocates of the Modular Development for the building and construction trades. Their program conceives adherence to governing dimensions of 4", 16" and 40". The deliberate disestablishment of the foot and the yard in favor of dimensions tending to conform to metric sizes should arouse the ire of every fervent dodekaphile. It seems to us to be greater common sense to urge conformance to sizes of 4", 12", and 36", which fit the long established and world known standards. Duodecimally, these sizes are *4*", *10*" and *30*", and if it should be thought preferable to have the *4:10* ratio predominate, there is every reason to sanction *4*", *10*" and *40*" (or *4* feet).

. . . Vice-President Paul E. Friedemann has been giving much thought to improvement in the use of the particular talents of our members for the progress of the Society. The appeal of duodecimals acts as a sort of differential screen for the self selection of people with unusual abilities. The usual processes of putting these abilities into operation do not well fit this unusual condition. And he seeks to devise a convenient method of (a) discovering and cross indexing our talents, and (b) effecting their comfortable and rewarding application.

. . . This is a difficult undertaking; no complete solution can be expected. But through his effort some improvement in our organization may be attainable, and any result is pure profit. If your own thoughts develop fresh approaches to any phase of the problem, please write to Mr. Friedemann. He will probably have to write to you anyway, and his letter can be an answer to your suggestion.

Ye Ed.

ANSWERS TO CRYPTARITHMS

Last Issue	This Issue
Why education	Almond jets
Two manicures	Dye chromanil
	Plumbago dyes
	Negrito black

Our common number system is decimal - based on ten. The dozen system uses twelve as the base. This requires two additional symbols: X, called *dek*, is used for ten, and £, called *el*, is used for eleven. Twelve is written *10*, and is called *do*, for dozen. The quantity *one gross* is written *100*, and is called *gro*. *1000* is called *mo*, representing the meg-gross, or great-gross.

Modern numeration employs one of the greatest of man's inventions, the zero - symbol for nothing. It permits the use of place values. In our customary counting, the places in our numbers represent successive powers of ten; that is, in 365, the 5 applies to units, the 6 applies to tens, and the 3 applies to tens-of-tens, or hundreds. Place value is even more important in dozenal counting. For example, *265* represents *5* units, *6* dozen, and *2* dozen-dozen, or gross. This number would be called *2 gro 6 do 5*, and by a coincidence, represents the same quantity normally expressed as 365.

Place value is the whole key to dozenal arithmetic. Observe the following additions, remembering that we add up to a dozen before carrying one.

94	*136*	Five ft. nine in.	*5.9'*
31	*694*	Three ft. two in.	*3.2'*
96	*3£2*	Two ft. eight in.	*2.8'*
19£	*1000*	Eleven ft. seven in.	*£.7'*

You will not have to learn the dozenal multiplication tables since you already know the 12-times table. Mentally convert the quantities into dozens, and set them down. For example, 7 times 9 is 63, *which is* 5 dozen and 3; so set down *53*. Using this "*which is*" step, you will be able to multiply and divide dozenal numbers without referring to the dozenal multiplication table.

Conversion of small quantities is obvious. By simple inspection, if you are 35 years old, dozenally you are only *2£*, which is two dozen and eleven. For larger numbers, keep dividing by 12, and the successive remainders are the desired dozenal number.

```
12 ) 365
  12 ) 30  + 5
    12 ) 2  + 6
         0  + 2     Answer: 265
```

Dozenal numbers may be converted to decimal numbers by setting down the units figure, adding to it 12 times the second figure, plus 12^2 (or 144) times the third figure, plus 12^3 (or 1728) times the fourth figure, and so on as far as needed. Or, to use a method corresponding to the illustration, keep dividing by X, and the successive remainders are the desired decimal number.

Fractions may be similarly converted by using successive multiplications, instead of divisions, by 12 or X.

Numerical Progression

1	One		
10	Do	*.1*	Edo
100	Gro	*.01*	Egro
1,000	Mo	*.001*	Emo
10,000	Do-mo	*.000,1*	Edo-mo
100,000	Gro-mo	*.000,01*	Egro-mo
1,000,000	Bi-mo	*.000,001*	Ebi-mo
1,000,000,000	Tri-mo	and so on.	

Multiplication Table

1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*X*	*£*
2	*4*	*6*	*8*	*X*	*10*	*12*	*14*	*16*	*18*	*1X*
3	*6*	*9*	*10*	*13*	*16*	*19*	*20*	*23*	*26*	*29*
4	*8*	*10*	*14*	*18*	*20*	*24*	*28*	*30*	*34*	*38*
5	*X*	*13*	*18*	*21*	*26*	*2£*	*34*	*39*	*42*	*47*
6	*10*	*16*	*20*	*26*	*30*	*36*	*40*	*46*	*50*	*56*
7	*12*	*19*	*24*	*2£*	*36*	*41*	*48*	*53*	*5X*	*65*
8	*14*	*20*	*28*	*34*	*40*	*48*	*54*	*60*	*68*	*74*
9	*16*	*23*	*30*	*39*	*46*	*53*	*60*	*69*	*76*	*83*
X	*18*	*26*	*34*	*42*	*50*	*5X*	*68*	*76*	*84*	*92*
£	*1X*	*29*	*38*	*47*	*56*	*65*	*74*	*83*	*92*	*X1*